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USING THE LAGRANGE EQUATION TO SOLVE MECHANICS PROBLEMS INVOLVING CONSTRAINTS AND SUPPORTS

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Abstract
This article examines the theoretical and practical aspects of solving mechanics problems involving constraints and supports using the Lagrange equation. Compared with traditional Newtonian methods, the approach based on the Lagrange formalism allows one to derive equations of motion, determine the reaction forces of constraints, and analyze representative systems such as the pendulum, half-Atwood machine, yo-yo, spherical surface, and inclined plane. The approach helps students develop theoretical thinking, a deeper understanding of mechanical phenomena, and the ability to solve problems by diverse analytical methods.

Keywords: Mechanics; equation of motion; Lagrangian function; Lagrange equation; generalized coordinate; principle of least action; action integral; partial derivative; constraint; reaction force.

Introduction

Аннотация
В статье рассматриваются теоретические и практические аспекты решения задач механики с наличием связей и опор с использованием уравнения Лагранжа. В сравнении с традиционными методами на основе законов Ньютона показано, что применение лагранжевой формализма позволяет выводить уравнения движения, определять силы реакции связей и анализировать типичные системы — маятник, полуатвудову машину, йо-йо, сферическую поверхность и наклонную плоскость. Предложенный подход способствует развитию теоретического мышления студентов, глубокому пониманию механических явлений и формированию навыков решения задач различными методами.

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Ключевые слова: механика; уравнение движения; функция Лагранжа; уравнение Лагранжа; обобщённая координата; принцип наименьшего действия; интеграл действия; частная производная; связь; сила реакции.

It is well known that in general-education schools, academic lyceums, and in higher education programs other than physics, the mechanics portion of physics is studied mainly based on Newtonian mechanics and Newton's laws. When analyzing the motion of a body in the presence of constraints, one applies Newton's laws and determines either the reaction force of the support that restricts the motion or the tension forces in the constraints. If the motion of a body occurs along the Ox axis while motion along Oy is constrained, then according to Newton's laws, the projection of the forces along the direction of motion is $\sum F_{i.x} = ma_x$, whereas the projection along the constrained direction is $\sum F_{i.y} = 0$. Using these equations, one can determine unknown quantities in the constraints and the reaction or tension forces. Students at school and most university students develop the habit of solving such mechanics problems solely by Newton's laws. However, they have hardly any skill in solving them by theoretical means—that is, using the formulas of Lagrange and Hamilton. It is therefore timely to explain mechanics problems involving constraints and supports by theoretical methods and to teach students to work with the Lagrange or Hamilton equations, forming the corresponding knowledge and skills. This article is devoted to addressing this issue.

A constraint is a cause that restricts the motion of a body, such as a support or a string. A free system consisting of N particles has, as we know, a total of 3N degrees of freedom. If k of those particles are subject to constraints, the number of degrees of freedom is reduced accordingly by k. From the fact that the distance between constrained points remains constant, one can derive constraint equations. For example, for a system of two bodies, one can write constraint equations of the form.

$\left|\vec{r}_2(t) - \vec{r}_1(t)\right| = \ell$ or of the form

$$\left(x_2(t) - x_1(t)\right)^2 + \left(y_2(t) - y_1(t)\right)^2 + \left(z_2(t) - z_1(t)\right)^2 = \ell^2$$

For a point moving on the surface of a sphere of radius R centered at a given point, the constraint equation can be written as $\left|\vec{r}(t) - \vec{r}_0\right| = R$ or as

$(x(t)-x_0)^2+(y(t)-y_0)^2+(z(t)-z_0)^2=R^2$ And so on—many similar examples can be given.

Constraints can themselves be divided into several types. In the most general case, when the constraint equation depends on velocities as well as on coordinates and time, the constraint is called nonholonomic or non-integrable. A nonholonomic constraint equation has the form.

$$f_i\left(\dot{\vec{r}}_1,\dot{\vec{r}}_2,\ldots,\dot{\vec{r}}_N,\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N,t\right)=0,\quad (i=1,2,\ldots k) \qquad \textbf{(I)}$$

and belongs to the class of equations that cannot be integrated. By contrast, constraints that can be integrated are called holonomic, and it is with such equations that one usually works in practice. If velocity does not enter the constraint equation, i.e., if the constraint has the form

$$f_i\left(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N,t\right)=0,\quad (i=1,2,\ldots k) \qquad \textbf{(II)}$$

Then it is a holonomic constraint. Holonomic constraints are, in turn, divided into two types: (1) scleronomic—holonomic constraints that do not depend on time; and (2) rheonomic—holonomic constraints that do depend on time.

Although velocity does not appear explicitly in a holonomic constraint, such constraints still restrict the velocity in a certain direction. To see this, differentiate equation (II) with respect to time to obtain

$$\sum_{j=1}^{N}\frac{\partial f_i}{\partial r_j}\dot{\vec{r}}_j+\frac{\partial f_i}{\partial t}=0,\quad (i=1,2,\ldots k) \qquad \textbf{(III)}$$

A holonomic constraint also restricts acceleration. To see this, differentiate (III) once more with respect to time to obtain

$$\sum_{j=1}^{n}\frac{\partial f_i}{\partial r_j}\ddot{\vec{r}}_j+\sum_{j=1}^{n}\frac{\partial^2 f_i}{\partial r_k\partial r_j}\dot{\vec{r}}_j\dot{\vec{r}}_k+\frac{\partial^2 f_i}{\partial t^2}=0,\quad (i=1,2,\ldots k) \qquad \textbf{(IV)}$$

As is known, force and acceleration are proportional to one another (for a given mass). Therefore, constraints may be regarded as additional forces (support reaction forces) acting between the parts of a system.

If we express the holonomic constraint (II) in a simpler form as

$$f_i(q,t)=0,\quad (i=1,2,\ldots k) \qquad \textbf{(V)}$$

Then the action integral for a constrained body acquires an extra term compared with that of a free body,

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$$S' = S + \sum_{i=1}^{n} \int_{t_a}^{t_b} \lambda_i f_i(q,t)\,dt = \int_{t_a}^{t_b} \left[L(q,\dot{q},t) + \sum_{i=1}^{n} \lambda_i f_i(q,t) \right] dt \qquad \textbf{(VI)}$$

where λ is the Lagrange multiplier.

Similarly, the Lagrangian function for the constrained body also differs from that of the free body by an additional term,

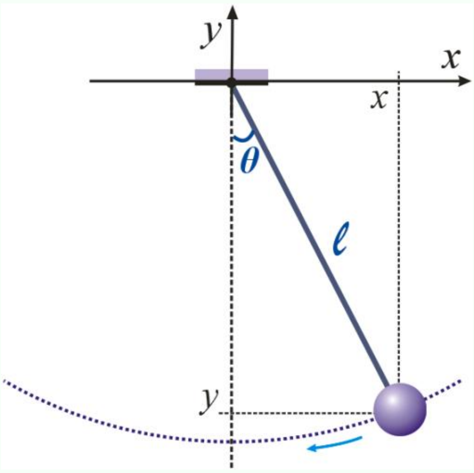
$$L' = L + \sum_{i=1}^{n} \lambda_i f_i(q,t) \qquad \textbf{(VII)}$$

Applying the action principle to the constrained body yields the Euler–Lagrange equation.

$$\frac{\partial L}{\partial q_i} + \lambda_i \frac{\partial f_i}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \quad \text{or} \quad \frac{\partial L}{\partial q_i} + \lambda_i \frac{\partial f_i}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \qquad \textbf{(VIII)}$$

From (VIII) it follows that the Euler–Lagrange equation for a constrained body differs from that of a free body by only a single term. Using (VIII), one can determine the reaction forces in a wide variety of problems concerning bodies or systems of bodies with constraints.

Let us now select several typical dynamical problems in mechanics that involve constraints and, for each, determine the corresponding reaction forces using the Lagrange equation.



Problem 1 (pendulum suspended from the ceiling). One end of a string of length ℓ is attached to the ceiling; a mass m is attached to the other end. If the mass is displaced to an arbitrary angle and released, it oscillates. Using the Lagrange equation, derive the dependence of the string tension on the angular displacement for the pendulum at an arbitrary position.

Solution. Take the suspension point as the origin and introduce axes Ox and Oy (see the figure). Since the pendulum moves in the Oxy plane, the mass has two degrees of freedom. There is, however, one constraint—the mass is connected to the ceiling by the string—so a single scleronomic constraint equation is written.

$$x^2 + y^2 = \ell^2 \qquad \textbf{(1.1)}$$

From this, the constraint equation f(x, y) can be written in the form

$$f(x,y) = \ell^2 - x^2 - y^2 = 0 \qquad \textbf{(1.2)}$$

When the pendulum is at an arbitrary position, it makes an angle θ with the vertical. The coordinates and velocities are then.

$$\begin{cases} x = \ell \sin\theta \\ y = -\ell\cos\theta \end{cases} \qquad \textbf{(1.3)}$$

The kinetic and potential energies are

$$K = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \qquad \textbf{(1.4)}$$

$$U = mgy \qquad \textbf{(1.5)}$$

Hence, for the oscillating simple pendulum, the Lagrangian, in accordance with (VII), is

$$L = K - U = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy + \lambda\left(\ell^2 - x^2 - y^2\right) \qquad \textbf{(1.6)}$$

Compute the partial derivatives required in the Lagrange equation:

$$\begin{cases} \dfrac{\partial L}{\partial q_1} = \dfrac{\partial L}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy + \lambda\left(\ell^2 - x^2 - y^2\right)\right) = -2\lambda x \\ \dfrac{\partial L}{\partial q_2} = \dfrac{\partial L}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy + \lambda\left(\ell^2 - x^2 - y^2\right)\right) = -mg - 2\lambda y \end{cases} \qquad \textbf{(1.7)}$$

$$\begin{cases} \dfrac{\partial L}{\partial \dot{q}_1} = \dfrac{\partial L}{\partial \dot{x}} = \dfrac{\partial}{\partial \dot{x}}\left(\dfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy + \lambda\left(\ell^2 - x^2 - y^2\right)\right) = m\dot{x} \\ \dfrac{\partial L}{\partial \dot{q}_2} = \dfrac{\partial L}{\partial \dot{y}} = \dfrac{\partial}{\partial \dot{y}}\left(\dfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy + \lambda\left(\ell^2 - x^2 - y^2\right)\right) = m\dot{y} \end{cases} \qquad \textbf{(1.8)}$$

$$m\ddot{x} = -2\lambda x \qquad \textbf{(1.9)}$$

$$m\ddot{y} = -mg - 2\lambda y \qquad \textbf{(1.10)}$$

Next, take the first and second time derivatives of (1.3):

$$\begin{cases} \dot{x} = \ell\dot{\theta}\cos\theta \\ \dot{y} = \ell\dot{\theta}\sin\theta \end{cases} \text{ and } \begin{cases} \ddot{x} = -\ell\dot{\theta}^2\sin\theta + \ell\ddot{\theta}\cos\theta \\ \ddot{y} = \ell\dot{\theta}^2\cos\theta + \ell\ddot{\theta}\sin\theta \end{cases} \qquad \textbf{(1.11)}$$

Substitute the second derivatives from (1.11) into (1.9) and (1.10) to obtain

$$m\ell\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) + 2\lambda\ell\sin\theta = 0 \qquad \textbf{(1.12)}$$

$$m\ell\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) - 2\lambda\ell\cos\theta = -mg \qquad \textbf{(1.13)}$$

Multiplying (1.12) by cosθ and (1.13) by sinθ and adding yields the differential equation determining θ(t):

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$$m\ell\ddot{\theta} = -mg\sin\theta, \quad \rightarrow \quad \ddot{\theta} + \frac{g}{\ell}\sin\theta = 0 \qquad \textbf{(1.14)}$$

Multiplying (1.12) by sinθ and (1.13) by cosθ and subtracting yields a differential equation for λ:

$$-m\ell\dot{\theta}^2 + 2\lambda\ell = mg\cos\theta, \quad \rightarrow \quad \lambda = \frac{1}{2}m\dot{\theta}^2 + \frac{1}{2\ell}mg\cos\theta \qquad \textbf{(1.15)}$$

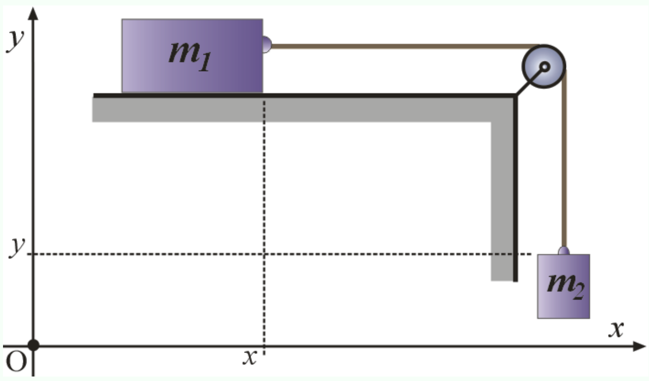
Clearly, λ is not an independent quantity; its solution is completely determined by $\theta(t)$, i.e., by (1.14). The quantity $2\lambda\ell$ has the dimensions of force and can be interpreted as the string tension,

$$T = 2\lambda\ell = m\dot{\theta}^2\ell + mg\cos\theta \qquad \textbf{(1.16)}$$

Thus, the tension in the string depends on time as in (1.16).

Problem 2 (Atwood's machine). A body of mass m_1 is on a smooth horizontal surface, and a body of mass m_2 is attached to it by a string and hangs vertically; this arrangement is called an Atwood's machine. Use the Lagrange equation to determine the tension in the string.

Solution. The arrangement is shown in the figure. Choose an arbitrary point in the plane of the figure as the origin and draw axes *Ox* and *Oy* through it. In this coordinate system, m_1 has coordinate x and m_2 has coordinate y. When the bodies are released, x increases while y decreases, but the sum of the coordinates does not change with time, i.e.,



$$x + y = C \qquad \textbf{(2.1)}$$

or, as a function,

$$f(x,y) = x + y - C = 0 \qquad \textbf{(2.2)}$$

Differentiating (2.1) with respect to time gives

$$\begin{cases} \dot{x} = -\dot{y} \\ \ddot{x} = -\ddot{y} \end{cases} \qquad \textbf{(2.3)}$$

There are two generalized coordinates in this case, so we write two Lagrange equations,

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$$\begin{cases} \dfrac{\partial L}{\partial x} + \lambda \dfrac{\partial f}{\partial x} = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}} \\ \dfrac{\partial L}{\partial y} + \lambda \dfrac{\partial f}{\partial y} = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{y}} \end{cases}, \text{ where } \begin{cases} F_x = \lambda \dfrac{\partial f}{\partial x} \\ F_y = \lambda \dfrac{\partial f}{\partial y} \end{cases} \quad \textbf{(2.4)}$$

The kinetic part of the Lagrangian for the system is

$$L = K - U = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + m_2 g y \quad \textbf{(2.5)}$$

Compute the partial derivatives,

$$\begin{cases} \dfrac{\partial L}{\partial x} = 0, \quad \dfrac{\partial f}{\partial x} = 1, \quad \dfrac{\partial L}{\partial \dot{x}} = m_1\dot{x} \\ \dfrac{\partial L}{\partial y} = m_2 g, \quad \dfrac{\partial f}{\partial y} = 1, \quad \dfrac{\partial L}{\partial \dot{y}} = m_2\dot{y} \end{cases} \quad \textbf{(2.6)}$$

and substitute them into the Lagrange equations,

$$\lambda = m_1\ddot{x} \quad \textbf{(2.7)}$$

$$m_2 g + \lambda = m_2\ddot{y} \quad \textbf{(2.8)}$$

Substituting (2.8) into (2.7) and using (2.3) yields

$$m_2 g + m_1\ddot{x} = -m_2\ddot{x}, \ \rightarrow \ m_2 g = -(m_1 + m_2)\ddot{x}, \ \rightarrow \ \ddot{x} = -\frac{m_2}{m_1 + m_2} g \quad \textbf{(2.9)}$$

$$\ddot{y} = -\ddot{x} = \frac{m_2}{m_1 + m_2} g \quad \textbf{(2.10)}$$

$$\lambda = m_1\ddot{x} = -\frac{m_1 m_2}{m_1 + m_2} g \quad \textbf{(2.11)}$$

Now determine the string tension,

$$\begin{cases} F_x = \lambda \dfrac{\partial f}{\partial x} = -\dfrac{m_1 m_2}{m_1 + m_2} g \cdot 1 = -\dfrac{m_1 m_2}{m_1 + m_2} g \\ F_y = \lambda \dfrac{\partial f}{\partial y} = -\dfrac{m_1 m_2}{m_1 + m_2} g \cdot 1 = -\dfrac{m_1 m_2}{m_1 + m_2} g \end{cases} \quad \textbf{(2.12)}$$

From (2.12), one can obtain the following special cases,

$$\begin{cases} m_1 >> m_2 \quad da \quad \lambda = -m_2 g \\ m_1 << m_2 \quad da \quad \lambda = -m_1 g \end{cases} \quad \textbf{(2.13)}$$

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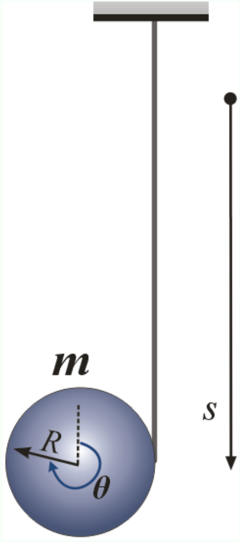

Problem 3 (yo-yo). A system in which a string is wound on a sphere or cylinder and the free end is held and driven to oscillate back and forth is called a yo-yo. Determine the tension in the string during oscillations using the Lagrange equation. In this problem, assume the string is attached to a homogeneous solid cylinder (spool). The radius of the cylinder is R and its mass is m.

Solution. Take the fixed end of the string as the origin and direct the Oy axis upward. If the spool descends, denote by s the length of the unwound portion of the string; then the vertical coordinate of the spool is $y = -s$ (see the figure). Then

$$R\theta = s \qquad \textbf{(3.1)}$$

or, as a function,

$$f(s,\theta) = R\theta - s = 0 \qquad \textbf{(3.2)}$$

Differentiating (3.1) with respect to time gives

$$\begin{cases} R\dot{\theta} = \dot{s} \\ R\ddot{\theta} = \ddot{s} \end{cases} \qquad \textbf{(3.3)}$$

There are two generalized coordinates in this case, so we write two Lagrange equations,

$$\begin{cases} \dfrac{\partial L}{\partial s} + \lambda \dfrac{\partial f}{\partial s} = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{s}} \\ \dfrac{\partial L}{\partial \theta} + \lambda \dfrac{\partial f}{\partial \theta} = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}} \end{cases}, \quad \text{where} \quad \begin{cases} F_s = \lambda \dfrac{\partial f}{\partial s} \\ F_\theta = \lambda \dfrac{\partial f}{\partial \theta} \end{cases} \qquad \textbf{(3.4)}$$

The kinetic part of the Lagrangian for the system is

$$L = K - U = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}I\dot{\theta}^2 - mgy = \frac{1}{2}m\dot{s}^2 + \frac{1}{4}mR^2\dot{\theta}^2 + mgs \qquad \textbf{(3.5)}$$

Compute the partial derivatives,

$$\begin{cases} \dfrac{\partial L}{\partial s} = mg, \quad \dfrac{\partial f}{\partial s} = -1, \quad \dfrac{\partial L}{\partial \dot{s}} = m\dot{s} \\ \dfrac{\partial L}{\partial \theta} = 0, \quad \dfrac{\partial f}{\partial \theta} = R, \quad \dfrac{\partial L}{\partial \dot{\theta}} = \dfrac{1}{2}mR^2\dot{\theta} \end{cases} \qquad \textbf{(3.6)}$$

and substitute them into the Lagrange equations,

$$mg - \lambda = m\ddot{s}, \quad \to \quad \lambda = mg - m\ddot{s} \qquad \textbf{(3.7)}$$

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$$0+\lambda R=\frac{1}{2}mR^2\ddot{\theta},\ \rightarrow\ \lambda=\frac{1}{2}mR\ddot{\theta}=\frac{1}{2}m\ddot{s} \quad \textbf{(3.8)}$$

Setting (3.8) equal to (3.7) gives the differential equation for s; then, using (3.3), obtain the differential equation for θ,

$$mg-m\ddot{s}=\frac{1}{2}m\ddot{s},\quad \rightarrow\quad \ddot{s}=\frac{2}{3}g \quad \textbf{(3.9)}$$

$$\ddot{\theta}=\frac{\ddot{s}}{R}=\frac{2}{3}\frac{g}{R} \quad \textbf{(3.10)}$$

Next, use (3.7) or (3.8) to determine the Lagrange multiplier λ,

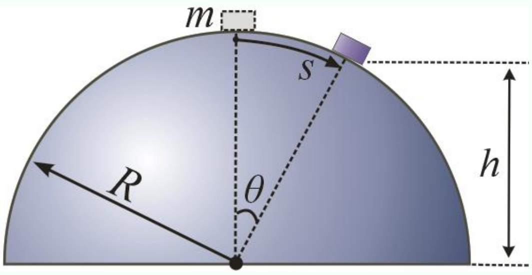
$$\lambda=\frac{1}{2}m\ddot{s}=\frac{1}{3}mg \quad \textbf{(3.11)}$$

Now determine the string tension,

$$\begin{cases} F_s=\lambda\dfrac{\partial f}{\partial s}=\dfrac{1}{3}mg\cdot(-1)=-\dfrac{1}{3}mg \\ F_\theta=\lambda\dfrac{\partial f}{\partial \theta}=\dfrac{1}{3}mg\cdot R=\dfrac{1}{3}mgR \end{cases} \quad \textbf{(3.12)}$$

Here, the tension T is the quantity of interest.

Problem 4 (separation of a body from a smooth spherical surface). A small body resting on the top of a smooth spherical surface of radius *R* receives a very small disturbance and begins to slide down the surface. At some point, the body loses contact due to the centrifugal inertial force and then continues its motion along a parabolic trajectory. Using the Lagrange equation, find the formula for the normal reaction force as a function of angle and the height of the separation point.



***Solution*.** The body slides along a hemispherical surface. Take the center of the base of the hemisphere as the origin and draw axes *Ox* and *Oy* through it. Let *(x, y)* be the coordinates of the body, *r* the distance to the origin, and *θ* the angle between r and the *Oy* axis (see figure). The Lagrangian of the body is then.

$$L=K-U=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)-mgy \quad \textbf{(4.1)}$$

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During the motion, *x, y, r,* and θ may all change. However, to avoid complicating the work, we aim to write as few Lagrange equations as possible. Therefore, it is more convenient to rewrite (4.1) in terms of r and θ,

$$\begin{cases} x = r\sin\theta \\ y = r\cos\theta \end{cases}; \quad \Rightarrow \quad \begin{cases} \dot{x} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\ \dot{y} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \end{cases} \qquad \textbf{(4.2)}$$

$$\begin{cases} \dot{x}^2 = \dot{r}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta + r^2\dot{\theta}^2\cos^2\theta \\ \dot{y} = \dot{r}^2\cos^2\theta - 2r\dot{r}\dot{\theta}\sin\theta\cos\theta + r^2\dot{\theta}^2\sin^2\theta \end{cases} \qquad \textbf{(4.3)}$$

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2 \qquad \textbf{(4.4)}$$

Substitute (4.4) into the Lagrangian (4.1),

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta \qquad \textbf{(4.5)}$$

There are two generalized coordinates in this case, so we write two Lagrange equations,

$$\begin{cases} \dfrac{\partial L}{\partial r} + \lambda\dfrac{\partial f}{\partial r} = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{r}} \\ \dfrac{\partial L}{\partial \theta} + \lambda\dfrac{\partial f}{\partial \theta} = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}} \end{cases}, \text{ where } \begin{cases} F_r = \lambda\dfrac{\partial f}{\partial r} \\ F_\theta = \lambda\dfrac{\partial f}{\partial \theta} \end{cases} \qquad \textbf{(4.6)}$$

As the holonomic constraint, we can take

$$f(r,\theta) = r - R \qquad \textbf{(4.7)}$$

While the body is on the sphere, *r-R=0*; otherwise *r-R>0*. Differentiating (4.7) under the condition *r-R=0* with respect to time gives

$$\begin{cases} \dot{r} = \dot{R} = 0 \\ \ddot{r} = \ddot{R} = 0 \end{cases} \qquad \textbf{(4.8)}$$

Compute the partial derivatives in the Lagrange equations (4.6) from the Lagrangian (4.5),

$$\begin{cases} \dfrac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\cos\theta, \quad \dfrac{\partial f}{\partial r} = 1, \quad \dfrac{\partial L}{\partial \dot{r}} = m\dot{r} \\ \dfrac{\partial L}{\partial \theta} = mgr\sin\theta, \quad \dfrac{\partial f}{\partial \theta} = 0, \quad \dfrac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \end{cases} \qquad \textbf{(4.9)}$$

and substitute them into the Lagrange equations,

$$mr\dot{\theta}^2 - mg\cos\theta + \lambda = m\ddot{r} \qquad \textbf{(4.10)}$$

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$$mgr\sin\theta = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} \qquad \textbf{(4.11)}$$

If the body moves without leaving the surface and taking (4.8) into account, from (4.10) and (4.11), we obtain

$$mR\dot{\theta}^2 - mg\cos\theta + \lambda = 0, \ \rightarrow \ \dot{\theta}^2 = \frac{g}{R}\cos\theta - \frac{\lambda}{mR}, \ \rightarrow \ \dot{\theta} = \sqrt{\frac{g}{R}\cos\theta - \frac{\lambda}{mR}}$$

(4.12)

$$mgR\sin\theta = mR^2\ddot{\theta}, \ \rightarrow \ \ddot{\theta} = \frac{g}{R}\sin\theta \qquad \textbf{(4.13)}$$

Taking the time derivative of (4.12) gives (4.13), so seek (4.12) in the general form.

$$\dot{\theta} = [A + B\cos\theta]^{\frac{1}{2}} \qquad \textbf{(4.14)}$$

Taking the time derivative of (4.14) and equating it to (4.13) yields the coefficient A,

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d}{dt}\left([A + B\cos\theta]^{\frac{1}{2}}\right) = -\frac{B\sin\theta\cdot\dot{\theta}}{2[A + B\cos\theta]^{\frac{1}{2}}} = -\frac{B}{2}\sin\theta.$$

$$\ddot{\theta} = -\frac{B}{2}\sin\theta = \frac{g}{R}\sin\theta, \ \rightarrow \ B = -\frac{2g}{R}\sin\theta \qquad \textbf{(4.15)}$$

Substituting (4.15) into (4.14) and applying the initial conditions

$$at \quad t = 0, \ \begin{cases} \theta = 0 \\ \dot{\theta} = 0 \end{cases} \qquad \textbf{(4.16)}$$

We obtain A completely,

$$0 = \left[A - \frac{2g}{R}\cos 0\right]^{\frac{1}{2}}, \ \rightarrow \ A = \frac{2g}{R} \qquad \textbf{(4.17)}$$

Substituting the coefficients from (4.15) and (4.17) into (4.14) yields the relation between angular speed and angle,

$$\dot{\theta} = \left[\frac{2g}{R} - \frac{2g}{R}\cos\theta\right]^{\frac{1}{2}} \qquad \textbf{(4.18)}$$

Comparing (4.18) with (4.12) gives the Lagrange multiplier λ,

$$\dot{\theta}^2 = \frac{2g}{R} - \frac{2g}{R}\cos\theta = \frac{g}{R}\cos\theta - \frac{\lambda}{mR}, \ \rightarrow \ \frac{\lambda}{mR} = \frac{3g}{R}\cos\theta - \frac{2g}{R}, \ \rightarrow$$

$$\lambda = (3\cos\theta - 2)mg \qquad \textbf{(4.19)}$$

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Since each term in (4.10)—including λ—has the dimensions of force, the Lagrange multiplier λ can be interpreted as the normal reaction force exerted by the spherical surface on the sliding body. When the body loses contact with the surface, $\lambda=0$. From this, we find the separation angle $\theta = \theta_0$,

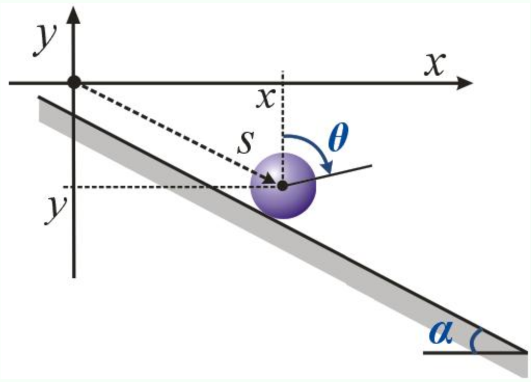
$$\lambda = (3\cos\theta_0 - 2)mg = 0, \quad \rightarrow \quad \cos\theta_0 = \frac{2}{3}, \quad \rightarrow \quad \theta_0 = \arccos\frac{2}{3} \approx 48^0 \qquad \textbf{(4.20)}$$

Now determine the height h of the separation point above the ground,

$$h = R - R\cos\theta_0 = R(1-\cos\theta_0) = R\left(1-\frac{2}{3}\right) = \frac{1}{3}R \qquad \textbf{(4.21)}$$

Thus, the reaction force of the spherical surface is given by the corresponding formula. $\lambda = (3\cos\theta - 2)mg$, and when $\theta_0 = \arccos\frac{2}{3} \approx 48^0$ If the angle or height satisfies the above conditions, the body leaves the surface.

Problem 5 (a body rolling down an inclined plane). Using the Lagrange equation, derive the formula for the reaction force exerted by a plane on a body rolling down an incline that makes an angle α with the horizontal.



Solution. Let the point at which the motion starts at the top of the incline be the origin, and from this point, denote by s the distance traveled along the plane (see figure). There is a constraint between the distance s and the rotation angle θ of the rolling body,

$$s = R\theta \qquad \textbf{(5.1)}$$

As the $f(s,\theta)$ holonomic constraint, we can take

$$f(s,\theta) = s - R\theta = 0 \qquad \textbf{(5.2)}$$

Differentiating (5.1) gives

$$\dot{s} = R\dot{\theta}, \quad \ddot{s} = R\ddot{\theta} \qquad \textbf{(5.3)}$$

The kinetic and potential energies are

$$K = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}I\dot{\theta}^2 \qquad \textbf{(5.4)}$$

$$U = mgy = -mgs\sin\alpha \qquad \textbf{(5.5)}$$

The Lagrangian for the rolling body is

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$$L = K - U = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}I\dot{\theta}^2 + mgs\sin\alpha \tag{5.6}$$

There are two generalized coordinates in this case, so we write two Lagrange equations,

$$\begin{cases} \dfrac{\partial L}{\partial s} + \lambda\dfrac{\partial f}{\partial s} = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{s}} \\ \dfrac{\partial L}{\partial \theta} + \lambda\dfrac{\partial f}{\partial \theta} = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}} \end{cases}, \text{ where } \begin{cases} F_s = \lambda\dfrac{\partial f}{\partial s} \\ F_\theta = \lambda\dfrac{\partial f}{\partial \theta} \end{cases} \tag{5.7}$$

Compute the partial derivatives of the Lagrangian (5.6),

$$\begin{cases} \dfrac{\partial L}{\partial s} = mg\sin\alpha, \quad \dfrac{\partial f}{\partial s} = 1, \quad \dfrac{\partial L}{\partial \dot{s}} = m\dot{s} \\ \dfrac{\partial L}{\partial \theta} = 0, \quad \dfrac{\partial f}{\partial \theta} = -R, \quad \dfrac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} \end{cases} \tag{5.8}$$

and substitute them into the Lagrange equations,

$$mg\sin\alpha + \lambda = m\ddot{s} \tag{5.9}$$

$$-\lambda \cdot R = I\ddot{\theta} \tag{5.10}$$

Taking (5.3) into account and performing algebraic transformations of (5.9) and (5.10) yields the Lagrange multiplier λ,

$$\begin{cases} mg\sin\alpha + \lambda = m\ddot{s} \\ -\lambda \cdot R = I\dfrac{\ddot{s}}{R} \end{cases}; \Rightarrow \begin{cases} mg\sin\alpha + \lambda = -\dfrac{\lambda mR^2}{I} \\ \ddot{s} = -\dfrac{\lambda R^2}{I} \end{cases}; \Rightarrow \lambda\left(1 + \frac{mR^2}{I}\right) = -mg\sin\alpha, \rightarrow$$

$$\lambda = -\frac{mg\sin\alpha}{1 + \dfrac{mR^2}{I}} \tag{5.11}$$

We can now determine the reaction force from (5.7),

$$F_s = \lambda\frac{\partial f}{\partial s} = -\frac{mg\sin\alpha}{1 + \dfrac{mR^2}{I}} = -\frac{mg\sin\alpha}{1+\beta} \tag{5.12}$$

Here $\beta = \dfrac{I}{mR^2}$ – It is a dimensionless coefficient less than 1 whose value is $\beta = \dfrac{1}{2}$ for a disk or solid cylinder, β = 1 for a ring or thin-walled tube, $\beta = \dfrac{2}{5}$ for

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a solid sphere, and $\beta = \frac{2}{3}$ For a spherical shell. Taking these values into account, formula (5.12) assumes the following forms:

$$F_s = -\frac{2}{3}mg\sin\alpha,\ F_s = -\frac{1}{2}mg\sin\alpha,\ F_s = -\frac{5}{7}mg\sin\alpha,\ F_s = -\frac{3}{5}mg\sin\alpha \quad (5.13)$$

Substituting the Lagrange multiplier (5.11) into (5.9) and (5.10) yields the equations of motion for the rolling body on the inclined plane. However, since these are not required by the problem statement, we do not derive them here.

Conclusion

In this article, we have demonstrated, through sample problems, how to solve mechanics problems involving supports and constraints not by the conventional method—Newton's laws—familiar to students, but by using the Lagrange equation and the Lagrangian function. Solving problems by this non-traditional method is very useful for students: it broadens their thinking, provides a more complete understanding of physical phenomena and processes, raises their level of knowledge, and develops skills for solving problems by different methods.

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