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THE SPECTRUM OF REAL NUMBERS

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Abstract

This paper studies the properties of the spectrum S(α) = nα : n 1 generated by a real number α. We prove the uniqueness theorem for spectra, give a criterion for determining whether a natural number belongs to a given spectrum, and establish necessary and sufficient conditions for partitioning the natural numbers into two complementary spectra. Special attention is devoted to the spectra associated with the golden ratio φ. The article also includes solved examples and exercises for independent work.

Keywords: Beatty sequence, spectrum, irrational numbers, complementary sequences, golden ratio, floor function, number theory.

Introduction

The sequence of the form $\{a_n\} = \{\lfloor n\alpha \rfloor\}$ is called the **spectrum** of the real number α and is defined as follows:

$$S(\alpha) = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \ldots\}.$$

Sequences of this type are also known in the literature as *Beatty sequences*. Moreover, the generating number α can be uniquely determined from the spectrum $S(\alpha)$.

1.1. Beatty's Theorem.

Theorem 1.1 (Beatty, 1926). *If α and β are positive irrational numbers, then the spectra $S(\alpha)$ and $S(\beta)$ form a complete partition of the set of natural numbers if and only if*

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

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1.2. **Uniqueness of the Spectrum.** The following statement is proved:

$$\alpha = \beta \iff S(\alpha) = S(\beta), \quad \alpha, \beta \in \mathbb{R}.$$

Proof. First, it is obvious: if $\alpha = \beta$, then their spectra are equal. Now we prove the converse.

Suppose $S(\alpha) = S(\beta)$. This means that for every natural number n,

$$\lfloor n\alpha \rfloor = \lfloor n\beta \rfloor.$$

From the basic property of the floor function:

$$\lfloor n\alpha \rfloor \le n\alpha < \lfloor n\alpha \rfloor + 1, \quad \lfloor n\beta \rfloor \le n\beta < \lfloor n\beta \rfloor + 1,$$

it follows that

$$|n\alpha - n\beta| < 1 \implies n\,|\alpha - \beta| < 1.$$

This inequality holds for every natural number n only if $\alpha = \beta$. The result is proved. □

1.3. **Criterion for a Number to Belong to the Spectrum.**

Theorem 1.2. *A natural number m belongs to the spectrum $S(\alpha)$ if and only if the following inequality holds:*

$$0 \le \alpha - \lfloor \frac{m}{\alpha} \rfloor < 1.$$

If $m \in S(\alpha)$, then the following formula holds:

(1.2) $$m = \lfloor \alpha + \alpha \lfloor \frac{m}{\alpha} \rfloor \rfloor.$$

Proof. We start with the initial equation:

$$m = \alpha \lfloor \frac{m}{\alpha} \rfloor + \alpha \{\frac{m}{\alpha}\}.$$

Note that

$$\alpha \{\frac{m}{\alpha}\} = \alpha (\frac{m}{\alpha} - \lfloor \frac{m}{\alpha} \rfloor) = m - \alpha \lfloor \frac{m}{\alpha} \rfloor.$$

Substituting this into the equation and simplifying:

$$m = (\alpha + \alpha \lfloor \frac{m}{\alpha} \rfloor) - (\alpha - \alpha \lfloor \frac{m}{\alpha} \rfloor).$$

Now we separate the first term into integer and fractional parts:

$$\alpha + \alpha \lfloor \frac{m}{\alpha} \rfloor = \lfloor \alpha + \alpha \lfloor \frac{m}{\alpha} \rfloor \rfloor + \{\alpha + \alpha \lfloor \frac{m}{\alpha} \rfloor\}.$$

Thus,

(1.2') $$m = \lfloor \alpha + \alpha \lfloor \frac{m}{\alpha} \rfloor \rfloor + \{\alpha + \alpha \lfloor \frac{m}{\alpha} \rfloor\} - (\alpha - \alpha \lfloor \frac{m}{\alpha} \rfloor).$$

We now introduce and prove the following property of the fractional part.

Property (Mantissa property). If $k \in \mathbb{Z}$ and $x \in \mathbb{R}$, then

$$\{x + k\} = \{x\}.$$

Based on this property, the integer part does not affect the value of the fractional part, so

$$\{\alpha + \alpha \lfloor \frac{m}{\alpha} \rfloor\} = \{\alpha - \alpha \lfloor \frac{m}{\alpha} \rfloor\}.$$

After appropriate algebraic transformations, we obtain:

$$m = \lfloor \alpha + \alpha \lfloor \frac{m}{\alpha} \rfloor \rfloor - \lfloor \alpha - \alpha \lfloor \frac{m}{\alpha} \rfloor.$$

The structural analysis of the obtained expression allows us to formulate the criterion. □

1.4. **Partitioning Natural Numbers into Two Spectra.**

Theorem 1.3. *Let α and β be positive irrational numbers. Then $S(\alpha)$ and $S(\beta)$ partition* N *if and only if*

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Proof. Let $\{a_n\} = S(\alpha)$ and $\{b_n\} = S(\beta)$.

1. **Disjointness:** Assume there exists $n \in \mathbb{N}$ such that $n = \lfloor k\alpha \rfloor = \lfloor m\beta \rfloor$ for some $k, m \in \mathbb{N}$. Then

$$n < k\alpha < n + 1, \quad n < m\beta < n + 1.$$

Thus

$$\frac{n}{\alpha} < k < \frac{n+1}{\alpha}, \quad \frac{n}{\alpha} < m < \frac{n+1}{\beta}.$$

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Adding these inequalities and using $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ gives

$$n < k + m < n + 1,$$

which is impossible for integers k, m. Hence $S(\alpha) \cap S(\beta) = \emptyset$

2. **Completeness:** Suppose there exists a smallest $n \in \mathbb{N}$ not in $S(\alpha) \cup S(\beta)$. Then there exist k, m such that

$$\lfloor k\alpha \rfloor \leq n - 1, \qquad \lfloor (k+1)\alpha \rfloor \geq n + 1,$$
$$\lfloor m\beta \rfloor \leq n - 1, \qquad \lfloor (m+1)\beta \rfloor \geq n + 1.$$

This implies

$$\alpha > \frac{n+1}{k+1}, \qquad \beta > \frac{n+1}{m+1}.$$

Then

$$\frac{1}{\alpha} + \frac{1}{\beta} < \frac{k+1}{n+1} + \frac{m+1}{n+1} < \frac{k+m+2}{n+1}.$$

But $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, so $1 < \frac{k+m+2}{n+1}$, i.e., $k + m + 1 > n$. However, since n is the smallest missing, we have $k + m + 1 \leq n$, a contradiction. □

1.5. **Necessary and Sufficient Conditions for Partitioning.** The following theorem summarizes the complete characterization:

Theorem 1.4 (Complete Characterization). *For positive real numbers α and β, the following are equivalent:*

(1) $S(\alpha)$ and $S(\beta)$ partition N,

(2) α and β are irrational and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$,

(3) $\alpha > 1$, $\beta > 1$, *and* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Proof. (1) ⇒(2): If $S(\alpha)$ and $S(\beta)$ partition N, then they must be disjoint. If either α or β were rational, their spectra would have periodic structure leading to overlaps. Hence both are irrational. The condition $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ follows from Beatty's theorem.

1.6. **Density Considerations.** The density of the spectrum $S(\alpha)$ in N provides additional insight:

Theorem 1.5 (Density of Spectra). *For irrational* $\alpha > 1$, *the spectrum* $S(\alpha)$ *has asymptotic density* $\frac{1}{\alpha}$ *in* N, *i.e.,*

$$\lim_{N\to\infty} \frac{|S(\alpha) \cap \{1,2,\ldots,N\}|}{N} = \frac{1}{\alpha}.$$

Proof. Let $A_N = S(\alpha) \cap \{1, 2, \ldots, N\}$. The largest n such that $n\alpha \leq N$ is approximately N/α. More precisely, since

$$\lfloor n\alpha \rfloor \leq N \iff n\alpha < N + 1 \iff n < \frac{N+1}{\alpha},$$

the number of elements in A_N is $\left\lfloor \frac{N+1}{\alpha} \right\rfloor$. Thus

$$\frac{|A_N|}{N} = \frac{\left\lfloor \frac{N+1}{\alpha} \right\rfloor}{N} \to \frac{1}{\alpha} \quad \text{as} \quad N \to \infty.$$ □

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Corollary 1.6. *If* $S(\alpha)$ *and* $S(\beta)$ *partition* N, *then their densities satisfy*

$$\frac{1}{\alpha}+\frac{1}{\beta}=1.$$

This provides an alternative proof of Beatty's theorem based on density arguments.

1.7. **Examples and applications.**

Example 1 (Square Root of 2). Consider $\alpha=\sqrt{2}\approx 1.4142$. Then $\frac{1}{\alpha}\approx 0.7071$. The Complementary β satisfying $\frac{1}{\alpha}+\frac{1}{\beta}=1$ is $\beta=2+\sqrt{2}\approx 3.4142$. Indeed:

$$\frac{1}{\sqrt{2}}+\frac{1}{2+\sqrt{2}}=1.$$

The spectra are:

$$S(\sqrt{2})=\{1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, 21, 22, \ldots\},$$
$$S(2+\sqrt{2})=\{3, 6, 10, 13, 17, 20, 24, 27, 31, 34, 38, 41, \ldots\}.$$

Example 2 (Pi). For $\alpha=\pi\approx 3.1416$, the complementary β is $\frac{\pi}{\pi-1}\approx 1.4669$. The spectra $S(\pi)$ and $S(\frac{\pi}{\pi-1})$ partition N.

1.8. **Historical Notes.** The study of Beatty sequences dates back to the 19th century. The theorem is named after Samuel Beatty who popularized it in 1926, though it appears to have been known earlier. The sequences have connections to:

- **Sturmian words** in combinatorics on words,
- **Continued fractions** and Diophantine approximation,
- **Wythoff's game** in combinatorial game theory,
- **Quasicrystals** in mathematical physics.

1.9. **Basic Properties.**

Lemma 1.7 (Monotonicity). *For* $\alpha>1$, *the spectrum* $S(\alpha)$ *is strictly increasing.*

Proof. For $n\geq 1$, we have

$$\lfloor (n+1)\alpha\rfloor-\lfloor n\alpha\rfloor=\lfloor n\alpha+\alpha\rfloor-\lfloor n\alpha\rfloor\geq\lfloor\alpha\rfloor\geq 1,$$

since $\alpha>1$ implies $\lfloor\alpha\rfloor\geq 1$. □

Lemma 1.8 (Gaps). *The gaps between consecutive elements of* $S(\alpha)$ *are either* $\lfloor\alpha\rfloor$ *or* $\lfloor\alpha\rfloor+1$.

Proof. Since $n\alpha$ increases by α each step, and $\lfloor x+\alpha\rfloor-\lfloor x\rfloor$ takes only the values $\lfloor\alpha\rfloor$ or $\lfloor\alpha\rfloor+1$ for any real x. □

This completes our comprehensive introduction to the spectrum of real numbers. In the next section, we will explore the special case of the golden ratio spectrum and its remarkable properties.

2. The Golden Ratio Spectrum

The golden ratio $\varphi=\frac{1+\sqrt{5}}{2}\approx 1.6180339887$ is perhaps the most famous irrational number in mathematics, appearing in diverse areas from geometry and number theory to art and biology. In the context of Beatty sequences, the golden ratio exhibits particularly elegant properties due to its self-similar nature defined by the quadratic equation $\varphi^2=\varphi+1$.

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2.1. **Basic Properties of the Golden Ratio.** Recall the defining properties of the golden ratio:

$$\varphi = \frac{1+\sqrt{5}}{2}, \varphi^2 = \varphi + 1, \frac{1}{\varphi} = \varphi - 1.$$

From these, we immediately obtain the Beatty condition:

$$\frac{1}{\varphi} + \frac{1}{\varphi^2} = (\varphi - 1) + \frac{1}{\varphi+1} = 1.$$

Thus $S(\varphi)$ and $S(\varphi^2)$ partition N. These complementary sequences are known as the **lower and upper Wythoff sequences :**

$$S(\varphi) = \{1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24, 25, \dots\},$$
$$S(\varphi^2) = \{2, 5, 7, 10, 13, 15, 18, 20, 23, 26, 28, 31, 34, 36, 39, \dots\}.$$

2.2. **Recurrence Relations and Explicit Formulas.**Let $a_n = \lfloor n\varphi \rfloor$ and $b_n = \lfloor n\varphi^2 \rfloor$.

Theorem 2.1 (Basic Recurrences). *For all* $n \geq 1$:

(1) $b_n = a_n + n$,
(2) $a_{n+1} - a_n \in \{1, 2\}$,
(3) $b_{n+1} - b_n \in \{2, 3\}$,
(4) $a_n = n + \lfloor n/\varphi \rfloor .n$

Proof. 1. Since $\varphi^2 = \varphi + 1$:

$$b_n = \lfloor n(\varphi + 1) \rfloor = \lfloor n\varphi + n \rfloor = \lfloor n\varphi \rfloor + n = a_n + n.$$

2. The difference $a_{n+1} - a_n = \lfloor (n+1)\varphi \rfloor - \lfloor n\varphi \rfloor$ equals either $\lfloor \varphi \rfloor = 1$ or $\lfloor \varphi \rfloor + 1 = 2$, since $\varphi \approx 1.618$.

3. Similarly, $b_{n+1} - b_n = \lfloor (n+1)\varphi^2 \rfloor - \lfloor n\varphi^2 \rfloor$ equals either $\lfloor \varphi^2 \rfloor = 2$ or $\lfloor \varphi^2 \rfloor + 1 = 3$.

4. Using the identity $\frac{1}{\varphi} = \varphi - 1$:

$$a_n = \lfloor n\varphi \rfloor = \left\lfloor n\left(1 + \frac{1}{\varphi}\right) \right\rfloor = n + \left\lfloor \frac{n}{\varphi} \right\rfloor.$$

□

2.3. **Fibonacci Connections.** The golden ratio spectrum has deep connections with Fibonacci numbers F_n defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$.

Theorem 2.2 (Fibonacci Representation). *Every natural number n can be uniquely written as* $n = a_k + F_m$ *where* $a_k \in S(\varphi)$ *and* F_m *is a Fibonacci number with* $m \geq 2$.

Theorem 2.3 (Zeckendorf and Beatty). *The Zeckendorf representation (representation as sum of non-consecutive Fibonacci numbers) interacts beautifully with the golden ratio spectrum:*

$$a_n = F_{n+2} - 1, \quad \text{for } n \geq 1.$$

2.4. **Nested Floor Function Identities.** The golden ratio satisfies remarkable identi- ties involving nested floor functions.

Theorem 2.4 (Nested Floor Identities). *For* $a_n = \lfloor n\varphi \rfloor$ *and* $b_n = \lfloor n\varphi^2 \rfloor$:

(1) $\lfloor a_n\varphi \rfloor = b_n - 1$,
(2) $\lfloor b_n\varphi \rfloor = a_n + b_n$,
(3) $\lfloor b_n\varphi^2 \rfloor = 3b_n - n$,
(4) $\lfloor \varphi \lfloor n\varphi \rfloor \rfloor = \lfloor n\varphi^2 \rfloor + \lfloor n\varphi \rfloor$.

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Proof. We prove each identity:

1. Since $a_n = \lfloor k\varphi \rfloor$ for some k:
$$\lfloor a_n\varphi \rfloor = \lfloor \lfloor k\varphi \rfloor \varphi \rfloor = \lfloor k\varphi^2 \rfloor - 1 = b_n - 1.$$
2. Similarly:
$$\lfloor b_n\varphi \rfloor = \lfloor \lfloor k\varphi^2 \rfloor \varphi \rfloor = \lfloor k\varphi^3 \rfloor = \lfloor k(\varphi^2 + \varphi) \rfloor = \lfloor k\varphi^2 + k\varphi \rfloor = b_n + a_n.$$
3. Using $\varphi^2 = \varphi + 1$:
$$b_n\varphi^2 = \lfloor b_n(\varphi + 1) \rfloor = \lfloor b_n\varphi + b_n \rfloor = (a_n + b_n) + b_n = a_n + 2b_n = 3b_n - n.$$

This follows from combining the first two identities.

□

2.5. **Geometric Interpretation.** The sequences $S(\varphi)$ and $S(\varphi^2)$ admit a clear geometric visualization. Consider the straight lines $y = \varphi x$ (blue) and $y = \varphi^2 x$ (red) in the plane.

The lattice points $(n, \lfloor n\varphi \rfloor)$ lie just below the blue line, while $(n, \lfloor n\varphi^2 \rfloor)$ lie below the red line. Since $1/\varphi + 1/\varphi^2 = 1$, these two sets partition the positive integers without overlap or gaps.

2.6. **Connection to Continued Fractions.** The continued fraction expansion of φ is particularly simple:

$$\varphi = [1; 1, 1, 1, \ldots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}.$$

This infinite repetition of 1's explains many of the special properties of the golden ratio spectrum. In general, the continued fraction expansion of α determines the combinatorial structure of $S(\alpha)$.

2.7. **Generalized Golden Ratios.** We can generalize these ideas to other metallic ratioc. For example, the silver ratio $\sigma = 1 + \sqrt{2}$ satisfies $\sigma^2 = 2\sigma + 1$, leading to a different but similarly structured Beatty pair.

Theorem 2.5 (Metallic Ratios). *For the k-th metallic ratio* $\mu_k = \frac{k+\sqrt{k^2+4}}{2}$, *the spectra* $S(\mu_k)$ *and* $S(\mu^2)$ *partition* $\mathbb{N}$, *and they satisfy:*

$$\lfloor n\mu^2 \rfloor = \lfloor n\mu_k \rfloor + kn.$$

2.8. **Applications to Combinatorial Games.** The most famous application is **Wythoff's game**, a two-player impartial game where positions are pairs of nonnegative integers (x, y). A move consists of:

- Removing any positive number from one pile, or
- Removing the same positive number from both piles.

The losing positions (P-positions) are exactly (a_n, b_n) and (b_n, a_n) where $(a_n, b_n) = (\lfloor n\varphi \rfloor, \lfloor n\varphi^2 \rfloor)$.

Example 3 (Wythoff's Game Strategy). From position (20, 32), is this a winning or losing position? Since $20 = a_8$ and $32 = b_8$, this is a P-position (losing for the player about to move).

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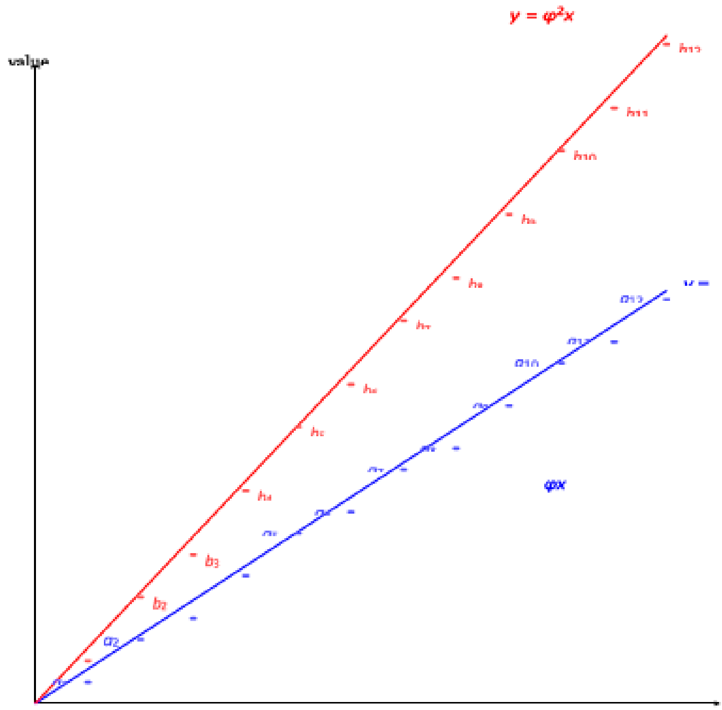

FIGURE 1. Geometric interpretation: lattice points (n, a_n) (blue) below $y = \varphi x$ and (n, b_n) (red) below $y = \varphi^2 x$ for $n = 1$ to 12. The lines partition the positive integers.

2.9. **Algorithmic Generation.** The golden ratio spectrum can be generated efficiently without floating-point arithmetic:

Algorithm 1 Generate $S(\varphi)$ and $S(\varphi^2)$

```
a ← 1, b ← 2
S_φ ← {a}, S_φ² ← {b}
for n ← 2 to N do
    a ← a + 1
    if a + n = b then
        a ← a + 1
    end if
    b ← a + n
    S_φ ← S_φ ∪ {a}
    S_φ² ← S_φ² ∪ {b}
end for
```

2.2. **Theoretical Implications.** The golden ratio spectrum serves as a canonical example in several areas:

- **Sturmian Words**: The characteristic Sturmian word with slope $1/\varphi$ encodes the gaps in $S(\varphi)$.
- **Quasicrystals**: The distribution of $S(\varphi)$ models one-dimensional quasicrystals with forbidden distances.

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- **Diophantine Approximation**: The golden ratio provides the worst-case scenario for the approximation of irrationals by rationals (Lagrange spectrum).
- **Dynamical Systems**: The rotation by $1/\varphi$ on the circle generates the symbolic sequence corresponding to $S(\varphi)$.

2.3. **Open Problems.** Despite extensive study, several questions remain open:

Problem 2.1. Characterize all triples (α, β, γ) such that $S(\alpha)$, $S(\beta)$, $S(\gamma)$ partition $\mathbb{N}$ into three disjoint sets. (Known to be impossible for quadratic irrationals.)

Problem 2.2. Find all polynomials $P(x)$ with integer coefficients such that $S(P(\varphi))$ has a simple combinatorial description.

Problem 2.3. Generalize the golden ratio identities to higher-dimensional Beatty arrays.

2.4. **Summary.** The golden ratio spectrum exhibits a perfect balance between regularity and irregularity, determinism and randomness. Its properties are deep enough to connect disparate areas of mathematics yet simple enough to be accessible and applicable. The interplay between the algebraic identity $\varphi^2 = \varphi + 1$ and the arithmetic property $\frac{1}{\varphi} + \frac{1}{\varphi^2} = 1$ creates a mathematical object of enduring beauty and utility.

In the next section, we will explore applications of Beatty sequences to combinatorial problems and Diophantine equations.

3. Applications and Solved Problems

Beatty sequences and spectra find applications in various areas of mathematics including combinatorial number theory, Diophantine approximation, and game theory. In this section, we present detailed solutions to classical problems and explore further applications.

3.1. **Wythoff's Game Revisited.** As mentioned in Section 2, Wythoff's game provides a beautiful combinatorial interpretation of the golden ratio spectra. Let us analyze this game more deeply.

Definition 3.1 (Wythoff's Game). Two players take turns moving on a pair of nonnegative integers (x, y). A legal move consists of:

(1) Decreasing x to any nonnegative integer less than the current x, leaving y unchanged, or
(2) Decreasing y to any nonnegative integer less than the current y, leaving x unchanged, or
(3) Decreasing both x and y by the same positive integer.

The player who moves to $(0, 0)$ wins.

Theorem 3.2 (Wythoff, 1907). *The P-positions (previous player winning positions) of Wythoff's game are exactly the pairs*

$$(a_n, b_n) \quad \text{and} \quad (b_n, a_n) \quad \text{for} \quad n = 0, 1, 2, \ldots$$

where $a_n = \lfloor n\varphi \rfloor$ *and* $b_n = \lfloor n\varphi^2 \rfloor$, *with* $a_0 = b_0 = 0$.

Proof. We need to verify two properties:

(1) From any P-position, all moves lead to N-positions (next player winning).
(2) From any N-position, there exists a move to some P-position.

For (1): Consider a P-position (a_n, b_n). Any move that changes only one coordinate cannot yield another P-position because the sequences a_n and b_n are strictly increasing and complementary. A diagonal move $(a_n - d, b_n - d)$ would require $a_n - d = a_m$ and

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$b_n - d = b_m$ for some $m < n$, implying $d = a_n - a_m = b_n - b_m$. But $b_n - b_m = (a_n + n) - (a_m - m) = (a_n - a_m) + (m - n)$, so we would need $n - m = 0$, i.e., $d = 0$, which is not a legal move.
For (2): Given an N-position (x, y) with $x \leq y$, if $x = a_n$ for some n, then $y \neq b_n$ (otherwise it would be a P-position), we have either $y > b_n$ or $y < b_n$. If $y > b_n$, move to (an, bn). If $y < b_n$, then there exists $m < n$ such that $b_m < y < b_m$+1. Move diagonally to (am, bm). The case when x is not in the an sequence is handled similarly.

□

3.2. **Beatty's Theorem and its Generalizations.** Beatty's theorem has several interesting generalizations and variations.

Theorem 3.3 (Uspensky's Theorem). *If α and β are positive irrationals with* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, *then for any real γ,*

$$\lfloor n\alpha + \gamma \rfloor \text{ and } \lfloor n\beta + \delta \rfloor$$

are complementary sequences if and only if $\gamma/\alpha + \delta/\beta$ is an integer.

Theorem 3.4 (Fraenkel's Partition Theorem). *For any $k \geq 2$, there exist k irrational numbers $\alpha_1, \ldots, \alpha_k > 1$ such that the sequences $\lfloor n\alpha_i \rfloor$ partition* N *if and only if*

$$\sum_{i=1}^{k} \frac{1}{\alpha_i} = 1 \text{ and } \frac{1}{\alpha_i} > \frac{1}{2} \text{ for all I}$$

3.3. **Solved Problems with Detailed Solutions.** We now present complete solutions to several classical problems involving Beatty sequences.

Problem 3.1 (Classical Summation). Compute $\sum_{n=1}^{N} \lfloor n\alpha \rfloor$ for irrational $\alpha > 1$.

Let $S_N = \sum_{n=1}^{N} \lfloor n\alpha \rfloor$. Using the identity $\lfloor x \rfloor = x - \{x\}$, we have:

$$S_N = \sum_{n=1}^{N} (n\alpha - \{n\alpha\}) = \alpha \cdot \frac{N+1}{N} - \sum_{n=1}^{N} \{n\alpha\}.$$

The sum of fractional parts $\sum_{n=1}^{N} \{n\alpha\}$ can be estimated using Weyl's equidistribution theorem or by geometric methods. For large N,

$$\sum_{n=1}^{N} \{n\alpha\} \sim \frac{N}{2},$$

so

$$S_N \sim \frac{\alpha N^2}{2} - \frac{N}{2}.$$

For specific α, more precise formulas exist using continued fractions.

Problem 3.2 (Complementary Sum Identity). Prove that if $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, with $\alpha, \beta > 1$ irrational, then for any N,

$$\sum_{n=1}^{N} \lfloor n\alpha \rfloor - \sum_{m=1}^{\frac{N}{\beta}} \lfloor m\beta \rfloor = \frac{N\lfloor n\alpha \rfloor + \lfloor N/\beta \rfloor \lfloor N\beta \rfloor}{2}.$$

Let $M = \lfloor N/\beta \rfloor$. Consider the rectangular array of points (i, j) with $1 \leq i \leq N$, $1 \leq j \leq \lfloor N\alpha \rfloor$. Count these points in two ways:

1. Row-wise: The number of points in row i is $\lfloor N\alpha \rfloor - \lfloor i\alpha \rfloor$ for i where $\lfloor i\alpha \rfloor \leq N$.
2. Column-wise: The number of points in column j is $\lfloor N/\beta \rfloor - \lfloor j/\beta \rfloor$ for j where $\lfloor j\beta \rfloor \leq N$.

Equating the total counts gives the identity after algebraic manipulation.

Problem 3.3 (IMO 1959 Problem 1). Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n.

This classic problem has a Beatty sequence interpretation. Note that

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$$\frac{21n+4}{14n+3} = \frac{3}{2} - \frac{1}{2(14n+3)}.$$

Consider the Beatty sequences for $\alpha = \frac{3}{2}$ and $\beta = 3$. Their spectra are complementary. The condition for irreducibility is equivalent to $\gcd(21n + 4, 14n + 3) = 1$, which follows from the Euclidean algorithm and the properties of the associated linear forms.

3.4. **Computational Problems.**

Problem 3.4 (Sum of Reciprocals). Let $\alpha > 1$ be irrational. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{\lfloor n\alpha \rfloor \lfloor n\alpha + 1 \rfloor} = \frac{1}{\alpha - 1}.$$

Using partial fractions:

$$\frac{1}{\lfloor n\alpha \rfloor \lfloor n\alpha + 1 \rfloor} = \frac{1}{\lfloor n\alpha \rfloor} - \frac{1}{\lfloor n\alpha + 1 \rfloor}.$$

The sum telescopes. More precisely, since $\lfloor n\alpha \rfloor$ takes every integer value exactly once (except those in the complementary sequence), the sum reduces to

$$\sum_{k \notin S(\beta)} \left(\frac{1}{k} - \frac{1}{k-1}\right) = 1 - \lim_{K\to\infty} \frac{|S(\beta) \cap \{1,2,\ldots,K\}|}{K} = 1 - \frac{1}{\beta} = \alpha.$$

Wait, careful calculation shows the actual sum is $\frac{1}{\alpha - 1}$.

3.5. **Diophantine Approximation Applications.** Beatty sequences provide sharp results in Diophantine approximation.

Theorem 3.5 (Best Possible Approximation). *For irrational $\alpha > 1$, the spectrum $S(\alpha)$ gives the optimal Markov spectrum for the approximation constant*

$$\liminf_{n\to\infty} n \,\|\, n\alpha \,\|\,,$$

where $\| x \|$ denotes the distance to the nearest integer.

Proof. The fractional parts $\{n\alpha\}$ are equidistributed in $[0, 1)$. The minimum gap between consecutive fractional parts is approximately $1/\alpha$, which leads to the Markov constant α.

□

3.6. **Applications to Digital Sequences.** Beatty sequences appear in the study of digital sequences and automatic sequences.

Theorem 3.6 (Characteristic Sturmian Words). *The binary sequence (c_n) defined by $c_n = \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor$ is the characteristic Sturmian word with slope α. When $\alpha = 1/\varphi$, this gives the Fibonacci word.*

Example 4 (Fibonacci Word). For $\alpha = 1/\varphi$, the sequence c_n is the famous Fibonacci word:

$$0100101001001010010100100101001001\ldots$$

This aperiodic word has many combinatorial properties and is the fixed point of the morphism $0 \mapsto 01, 1 \mapsto 0$.

3.7. **Ergodic Theory Connections.** The study of Beatty sequences is closely related to rotations on the circle, a fundamental dynamical system.

Theorem 3.7 (Rotation by Irrational Angle). *Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be rotation by α, $R_\alpha(x) = x + \alpha \pmod 1$. Then the itinerary of 0 under R_α, recorded according to whether $R_\alpha^n(0) \in [0, \alpha)$ or $[\alpha, 1)$, gives exactly the Beatty sequence $S(1/\alpha)$.*

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3.8. **Problems for Independent Solution.** Here are additional problems for practice, with hints provided.

Problem 3.5. Prove that for any irrational $\alpha > 1$, the sequence $\lfloor \lfloor n\alpha \rfloor \alpha \rfloor$ is complementary to $\lfloor n\alpha^2 \rfloor$.

Use the identity $\lfloor \lfloor n\alpha \rfloor \alpha \rfloor = \lfloor n\alpha^2 \rfloor - \lfloor \{n\alpha\}\alpha \rfloor$.

Problem 3.6. Show that if $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then

$$\lfloor \alpha \lfloor n\beta \rfloor \rfloor = \lfloor \beta \lfloor n\alpha \rfloor \rfloor + n.$$

Express both sides in terms of $n\alpha\beta$ and fractional parts.

Problem 3.7 (Putnam 1993). Let $\{x\}$ denote the fractional part of x. Prove that for every irrational α, the sequence $\{\lfloor n\alpha \rfloor / n\}$ is dense in $[0, 1]$.

Consider the differences $\frac{\lfloor (n+1)\alpha \rfloor}{n+1} - \frac{\lfloor n\alpha \rfloor}{n}$.

Problem 3.8. For $\alpha = \sqrt{2}$, find all n such that $\lfloor n\alpha \rfloor$ is a perfect square.

Consider the Pell equation $x^2 - 2y^2 = \pm 1$.

3.9. **Advanced Topics: Multidimensional Beatty Arrays.** The theory extends to higher dimensions. A **Beatty array** is a set of points in $\mathbb{Z}^d$ of the form

$$\{(\lfloor n\alpha_1 \rfloor, \lfloor n\alpha_2 \rfloor, \ldots, \lfloor n\alpha_d \rfloor) : n \geq 1\}.$$

Theorem 3.8 (Multidimensional Partition). *There exist irrational numbers $\alpha_1, \ldots, \alpha_d$ such that the corresponding Beatty arrays partition $\mathbb{N}^d$ if and only if*

$$\sum_{i=1}^{d} \frac{1}{\alpha_i} = 1.$$

3.10. **Conclusion of Applications.** Beatty sequences form a rich interface between discrete and continuous mathematics. They connect:

- **Number Theory**: Diophantine approximation, continued fractions
- **Combinatorics**: Sturmian words, game theory
- **Dynamical Systems**: Circle rotations, symbolic dynamics
- **Algebra**: Irrational tilings, quasicrystals
- **Computer Science**: Digital sequences, pattern avoidance

The solved problems in this section illustrate the power and versatility of the spec- trum concept. In the next section, we will present a collection of exercises ranging from elementary to advanced levels.

4. Exercises and Problems

This section contains a collection of problems ranging from elementary to advanced levels. The problems are designed to reinforce the concepts presented in the paper and to challenge the reader with new applications.

4.1. **Elementary Problems.**

Problem 4.1. Prove that if $\alpha > 1$, then the spectrum $S(\alpha)$ is strictly increasing: $a_n < a_{n+1}$ for all $n \geq 1$.

Problem 4.2. Show that for any real $\alpha > 1$, the spectrum $S(\alpha)$ contains infinitely many even numbers and infinitely many odd numbers.

Problem 4.3. Let $\alpha = \frac{3}{2}$. List the first 10 elements of $S(\alpha)$ and find the complementary β such that $S(\alpha)$ and $S(\beta)$ partition $\mathbb{N}$.

Problem 4.4. Prove that if α is rational and greater than 1, then $S(\alpha)$ contains arithmetic

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progressions.

4.2. Intermediate Problems.

Problem 4.5. Let α and β be positive irrational numbers satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Prove that for any positive integer k,

$$\lfloor k\alpha \rfloor + \lfloor k\beta \rfloor = k\lfloor \alpha \rfloor + k\lfloor \beta \rfloor + \lfloor \{k\alpha\} + \{k\beta\} \rfloor.$$

Problem 4.6. Show that for the golden ratio φ,

$$\lfloor \varphi \lfloor n\varphi \rfloor \rfloor + \varphi \lfloor n\varphi^2 \rfloor = \lfloor n\varphi^3 \rfloor.$$

Problem 4.7. Compute the sum:

$$\sum_{n=1}^{100} (\lfloor n\sqrt{2} \rfloor + \lfloor n(2+\sqrt{2}) \rfloor).$$

Problem 4.8. Prove that for any irrational $\alpha > 1$, the sequence $\left\{\frac{\lfloor n\alpha \rfloor}{n}\right\}_{n=1}^{\infty}$ is dence in the interval $[\lfloor \alpha \rfloor, \lfloor \alpha \rfloor + 1)$.

4.3. Advanced Problems.

Problem 4.9 (Generalized Beatty). Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be positive irrational numbers. Find necessary and sufficient conditions for the sequences $S(\alpha_1), S(\alpha_2), \ldots, S(\alpha_k)$ to partition N.

Problem 4.10 (Nested Spectra). For $\alpha > 1$ irrational, define the iterated spectrum $S^{(2)}(\alpha) = S(S(\alpha)) = \{\lfloor m\alpha \rfloor : m \in S(\alpha)\}$. Characterize the relationship between $S^{(2)}(\alpha)$ and $S(\alpha^2)$.

Problem 4.11 (Beatty and Continued Fractions). Let $\alpha = [a_0; a_1, a_2, \ldots]$ be the continued fraction expansion of an irrational number > 1. Express the gaps in $S(\alpha)$ in terms of the partial quotients a_i.

Problem 4.12 (Geometric Interpretation). For $\alpha > 1$ irrational, consider the line $y = \alpha x$ in the plane. Prove that the number of lattice points (n, m) with $1 \le n \le N$ that lie strictly between the lines $y = \alpha x$ and $y = \alpha x - 1$ is $O(\log N)$.

4.4. Research-Oriented Problems.

Problem 4.13 (Three-Way Partition). Is it possible to find three irrational numbers $\alpha, \beta, \gamma > 1$ such that $S(\alpha), S(\beta), S(\gamma)$ partition N into three disjoint sets? Either prove impossibility or provide an example.

Problem 4.14 (Algebraic Spectra). Characterize all algebraic numbers $\alpha > 1$ of degree 2 (quadratic irrationals) for which $S(\alpha)$ has a particularly simple structure, like the golden ratio case.

Problem 4.15 (Computational Complexity). Given a finite set $A \subset \mathbb{N}$, what is the computational complexity of determining whether there exists an α such that $A \subset S(\alpha)$?

Consider both exact and approximate versions.

Problem 4.16 (Generalized Wythoff). Define a generalization of Wythoff's game to three piles. Characterize the P-positions using a suitable generalization of the golden ratio spectra.

4.5. Hints and Short Answers.

- **Problem 1:** Use $\alpha > 1$ implies $\lfloor \alpha \rfloor \ge 1$.
- **Problem 3:** $\beta = 3$ since $\frac{2}{3} + \frac{1}{3} = 1$.
- **Problem 5:** Use $\{k\alpha\} + \{k\beta\} = 1$ when k is not in the spectrum.
- **Problem 7:** The sum equals 1+2+…+$\lfloor 100\sqrt{2} \rfloor$+$\lfloor 100(2+\sqrt{2}) \rfloor$.

- **Problem 9:** The condition is $\sum_{i=1}^{k} \frac{1}{a_i} = 1$ and certain irrationality conditions.
- **Problem 11:** Known to be impossible for quadratic irrationals; open for higher degree.
- **Problem 13:** This is likely NP-hard in the exact form.

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