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ON THE CARLEMAN FORMULA IN SIEGEL DOMAINS

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Abstract:
This paper investigates the Carleman formulas within the framework of Siegel domains, which are prominent structures in the theory of several complex variables. Siegel domains, particularly of the second kind, provide a natural setting for the study of holomorphic functions and their boundary behaviors due to their geometric and analytical properties.

The Carleman formula serves as a reproducing integral formula, enabling the analytic continuation of functions from boundary subsets to the interior of the domain. In this study, we present generalized Carleman-type formulas adapted to the geometry of Siegel domains, incorporating Hermitian forms and differential structures. Special attention is given to the construction of integral kernels and the application of these formulas to classes of holomorphic functions

The results demonstrate how function values in a Siegel domain can be recovered from their boundary data, and how complex lines and automorphisms play a crucial role in the formulation. This contributes to a deeper understanding of boundary integral methods and the role of geometric symmetry in complex analysis.

Keywords. Siegel domain, Carleman formula, analytic continuation, reproducing kernel, boundary value problem, holomorphic function, several complex variables, Hermitian form, complex geometry, automorphism.

Introduction

An important role in the characterization of homogeneous bounded domains is played by the Siegel domains. Let V be a convex open pointed cone in $\mathbb{R}^n$ with the vertex at the origin. A function $F(u,v):\mathbb{C}^m\times\mathbb{C}^m\to\mathbb{C}^k$ is called a V-Hermitian non-degenerate form if:

1. $F(u,v)=\overline{F(v,u)}$,
2. F is linear in u,

3. $F(u,u)\in V$ and $F(u,u)=0$ if and only if $u=0$.

A Siegel domain of the second kind is a domain $D\subset C^n=C^k\times C^m$ of the form:

$$D=\left\{(z,u):\ z\in C^k,\ u\in C^m,\ \operatorname{Im} z - F(u,u)\in V\right\}. \tag{1}$$

Every Siegel domain of the form (1) is biholomorphically equivalent to a bounded domain. The fundamental theorem of Vinberg–Gindikin–Pyatetsky-Shapiro states the converse: every bounded homogeneous domain in C^n is biholomorphically equivalent to a homogeneous Siegel domain of the form (1).

The Shilov boundary of a Siegel domain of the form (1) is the set:

$$S=\left\{(z,u):\ \operatorname{Im} z=F(u,u)\right\},$$

It is a smooth generating manifold in C^n. Indeed, for every point $(z^0,u^0)\in S$, the complex tangent plane has the form:

$$T_S^c(z^0,u^0)=\left\{(z,u):\ z-z^0=2i\left[F(u,u^0)-F(u^0,u^0)\right]\right\},$$

The complex codimension of T_S^c equals k and it coincides with the real codimension of the submanifold S. Therefore, the manifold S is a real hypersurface of class C^∞ in C^n.

The Lebesgue measure μ on S is defined (up to a constant multiplier) by the differential form $\omega=dx\wedge du\wedge d\bar{u}$, where $x=\operatorname{Re} z$. Since D is biholomorphically equivalent to a bounded domain, we can define the class $A^*(D)\subset A(D)$ in the same way as in the case of the half-plane.

Let M be a closed bounded subset in S such that $\mu M>0$. According to Sadullaev's Theorem 0.7, M is a uniqueness set for $A^*(D)$. Therefore, just as in the classical case, the problem naturally arises of constructing Carleman formulas for the set M. However, unlike classical domains, in the Siegel domain D there are, strictly speaking, no points through which it is possible to conduct a sufficiently strong family of complex lines intersecting S only along curves.

Indeed, consider a complex line passing through the point $(z^0,u^0)\in D$ in the form:

$$\left\{(z,u):\ z=z^0+at,\ u=u^0+bt,\ a\in C^k,\ b\in C^m,\ t\in C^1\right\}$$

where $a\in C^k$, $b\in C^m$, and $t\in C^1$. Then the intersection with S is defined by the equation:

$$\operatorname{Im} z^0-F(u^0,u^0)=-\operatorname{Im} at+\bar{t}F(a^0,b)+tF(b,a^0)+|t|^2F(b,b),$$

In order for this intersection to be a curve (or straight line), one needs rigid constraints on a and b. For instance, if $b=0$, then a must satisfy:

$$a=\lambda[Imz^0-F(u^0,u^0)]$$

Therefore, the methods of do not automatically transfer to the Siegel domain.

We proceed as follows: first, we restore the value of the function f at one of the infinitely distant points of the set S, then, using automorphisms, we "spread" it over S, and then the function f can be analytically continued from S to the domain D using some integral representation for this domain

Let us consider a family of complex lines:

$$\begin{cases} \xi = at+z \\ \eta = u \end{cases}$$

where $(z,u)\in S$, $t\in \mathrm{C}^1$, and a is a point in the cone V. These lines intersect S along the line $\{t: Imt=0\}$, and the domain D corresponds to the half-plane $\Pi=\{t: Imt>0\}$. Indeed,

$$\operatorname{Im}\zeta - F(\eta,\eta)=\operatorname{Im}z-F(u,u)+\operatorname{Im}at=a\operatorname{Im}t$$

If $Imt>0$, then $aImt\in V$. Without loss of generality, we may assume that $a=(1,\ldots,1)\in V$ and $z=x+iy$, where $x=(0,x_2,\ldots,x_n)$. Then the Shilov boundary S can be decomposed into lines $l_{x,u}=\{(\xi,\eta):\xi=t+z,\eta=u,t\in \mathrm{R}^1\}$, where $(z,u)\in S$.

If a compact set $M\subset S$ satisfies $\mu M>0$, then define:

$$M_{'x,u}=M\bigcap l_{'x,u}, M^{'}=\left\{({}^{'}x,u):\ m_1 M_{'x,u}>0\right\}\subset R^{k-1}\times C^m$$

The set M' is measurable with respect to the measure m_{2m+k-1}, and by Fubini's theorem we have $m_{2m+k-1}M'>0$. The complex lines $l_{x,u}$ define a unique point at infinity on the compactification $\bar{S}$ of S in CP^{k+m}, which we denote by the symbol ∞.

Theorem 1. (Kytmanov–Nikitina). Let $f\in A^{\alpha}(D)$, $\alpha>0$. Then

$$f(\infty)=\frac{1}{2\pi i\int\limits_{M^{'}} d\mu_{'x,u}}\lim_{j\to\infty}\int\limits_{M}\frac{f(z,u)\varphi^{j}(z,u)d\mu_{z,u}}{z_1+i-iF(u,\vartheta)},$$

Let $\varphi=\exp\psi$, where

$$\psi(z,u)=\frac{i}{\pi}\int\limits_{M_{'x,u}}\frac{dt}{t-z_1+iF_1(u,u)}.$$

Proof. Consider the auxiliary function $F(t)=f(t+z,u)$, then $F\in A^{\alpha}(\Pi),$, hence

$$f(\infty)=F(\infty)=\frac{1}{2\pi i}\int_{M'_{x,u}}F(t)\varphi^{j}(t)\frac{dt}{t+i}$$

Note that

$$dz\wedge du\wedge d\bar{u}=dx\wedge du\wedge d\bar{u}=dt\wedge d'x\wedge du\wedge d\bar{u}.$$

Multiplying by $d'x\wedge du\wedge d\bar{u}$ and integrating over M', we obtain. The possibility of passing the limit under the integral sign over M' is justified by the fact that the integrals over $M_{x,u}$ are uniformly bounded for all j.

Now suppose the Siegel domain D is homogeneous. Every biholomorphic automorphism of D is a rational vector-function. Among all automorphisms, we can select affine ones, which form a transitive subgroup of automorphisms that act transitively on S. However, affine automorphisms do not map the point at infinity to a finite point. Therefore, using some automorphism, we first move our point at infinity ∞ to some finite point on S, and then we can apply affine automorphisms. In this way, we can map ∞ to any point in S.

The resulting automorphism generally maps special points in S, but the set of such points has zero measure (with respect to μ), since the manifold S is generated. Thus, we can assume (if necessary, by slightly shrinking M) that M does not intersect this set.

Let $(\Phi(z,u),\ \psi(z,u))-$ be an automorphism of D mapping a fixed point $(z^0,u^0)\in S\setminus M$ to the point ∞, where $\Phi=(\Phi_1,\ ...,\ \Phi_k),,\ \psi=(\psi_1,\ ...,\ \psi_m)$.

Theorem 2. (Kytmanov–Nikitina). If $f\in A^{\alpha}(D)$, then

$$f(z^0,u^0)=\frac{1}{C}\lim_{j\to\infty}\int_{M}f(z,u)\frac{g^{j}(z,u)d\Phi\wedge d\psi\wedge d\overline{\psi}}{\Phi_1(z,u)+i-iF_1(\psi,\psi)},$$

where

$$C=2\pi i\int_{M'}d'\Phi\wedge d\psi\wedge d\overline{\psi};\quad g(z,u)=\varphi(\Phi(z,u),\ \psi(z,u))$$

Proof. Follows directly from Theorem 1.

Example 1. Let

$$D=\left\{(z,u):\ z\in C^1,\ u\in C^m,\ \operatorname{Im}z-|u_1|^2-...-|u_m|^2>0\right\}$$

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This domain is biholomorphically equivalent to the unit ball. To explicitly write the kernel in formula for this case, we can biholomorphically map D to a ball B, perform a rotation in B, and then map back to D.

In doing so, we obtain an automorphism of the domain D that maps the unique point at infinity on the boundary S to any other point on S. Calculations show that for this D, the kernel in (26.4) takes the form:

$$\frac{i\left|z^0+i\right|^{2m+2}\wedge\quad\wedge d\bar{u}}{\rho(z,u,z^0,u^0)},$$

where

$$\rho(z,u,z^0,u^0)=\left|i(\bar{z}^0-z)-2<u,\bar{u}^0>\right|^{2m}\Big\{2\left|i(\bar{z}^0-z)-2<u,\bar{u}^0>\right|^2+$$
$$+i\Big[\left(\left|z\right|^2-1\right)\left(z^0+\bar{z}^0\right)-\left(\left|z^0\right|^2-1\right)\left(z+\bar{z}\right)+4i\left(z-i\right)<u,\bar{u}^0>\times\operatorname{Im}\left(\bar{z}^0-i\right)\Big]\Big\}.$$

Example 2. Let

$$D=\tau^+,$$

the **future tube** (see [57]), i.e., the tube domain over the future light cone:

$$\tau^+=\left\{z:\ z\in C^n,\ y_1^2>y_2^2+\ldots+y_n^2,\ \ y_1>0\right\}$$

The boundary S of the Siegel domain D is mapped into the real cone R^n (in this case, the cone V is Hermitian with respect to a quadratic form). In order to move the point at infinity to a fixed point in S, we use a transformation:

$$z\mapsto-\frac{(x/z)^2}{z^2},$$

and then apply inversion $z\mapsto z^{-1}$. As a result, the kernel in formula. takes the form:
Thus, for functions $f\in A^*(\tau^+)$ and points $x^0\in S$, the following formula holds:

$$f(x_0)=\frac{1}{C}\lim_{j\to\infty}\frac{f(x)\varphi_{x^0}^j(x)dx}{\left[(x-x^0)^2\right]^{n-1}\left[x_1-x_1^0+i(x-x^0)^2\right]},$$

where φ_{x_0} is obtained from φ after a corresponding change of variables. We can assume that $(x-x^0)^2\neq0$ on the set M.

In conclusion, let us discuss the question of how to reconstruct the values of f on the Shilov boundary S if the Siegel domain D is not homogeneous. Fix a point $(z^0,u^0)\in S$. Consider the complex lines passing through this point and intersecting the domain D. These lines fall into two classes:

1. $l_a = \left\{(z,u):\ z = at + z^0,\ u = u^0,\ t \in C^1\right\},\ a \in V \subset R^n,$ the set $l_a \cap D$ is a half-plane Π with respect to t.

2. $l_{b,p} = \left\{(z,u):\ z = 2i\left[F(b,u^0) - pF(b,b)\right]t + z^0,\ u = bt + u^0,\ t \in C^1\right\},$ $b \in \mathrm{C}^m, \rho \in \mathrm{C}^1, F(b,b) \in V$. In this case, $l_{b,\rho} \cap D$ is a disk of radius ρ centered at the point $t = -\bar{\rho}$.

Indeed, if $z = at + z^0, u = bt + u^0$, then:

$$\begin{aligned}\operatorname{Im} z - F(u,u) &= \operatorname{Im} z_0 + \operatorname{Im} at - F(u^0,u^0) - \bar{t}F(u^0,b) - |t|^2 F(b,b) = \\ &= \operatorname{Im} at - tF(b,u^0) - \bar{t}F(u^0,b) - |t|^2 F(b,b) \in V.\end{aligned}$$

So, if $F(b,b) = 0$, then we obtain lines of the first class. If $F(b,b) \neq 0$, then...

We have:

$$\operatorname{Im} a_j t - tF_j(b,u^0) - \bar{t}F_j(u^0,b) - |t|^2 F_j(b,b) = \qquad \text{for} \quad j = 1,2,\dots,k.$$

$$= F_j(b,b)\left[\left|t + \frac{F_j(u^0,b) - \frac{i}{2}\bar{a}_j}{F_j(b,b)}\right|^2 - \left|\frac{F(u^0,b) - \frac{i}{2}\bar{a}_j}{F_j(b,b)}\right|^2\right], \quad j = 1,2,\dots,k,$$

Therefore, for the domain D to intersect with such a line, it is necessary that $F(b,b) \in V$ and

$$\frac{F(u^0,b) - \frac{i}{2}\bar{a}_j}{F_j(b,b)} = p \quad j = 1,2,\dots,k$$

which corresponds to a second-class line.

Let $(z^0,u^0) \in S$ and the cone $V_{x^0,u^0} = \left\{(x,u^0):\ x = x^0 + \tilde{x},\ \tilde{x} \in V\right\}$ intersect M with a set of positive measure m_k. Then the value of $f \in A^{\alpha}(D)$ at (z^0,u^0) can be reconstructed as follows.

Consider the lines l_a of the first class; assume $a_1 = 1$. Noting that:

$$dx_1 \wedge \dots \wedge dx_k = t^{k-1} dt \wedge da_2 \wedge \dots \wedge da_k$$

we get:

$$\left(x_1 - x_1^0\right)^{-k} dx = t^{-1} dt \wedge da_2 \wedge \dots \wedge da^k,$$

Let $M_{x^0,u^0} = l_a \cap M$ and $M = \{a : a_1 = 1, m_1 M_a > 0\}$. Applying the method from the proof of Theorem 1, we obtain:

$$f(z^0,u^0) = \frac{1}{2\pi i \int\limits_{\tilde{M}} d^{'} a} \lim_{j\to\infty} \int\limits_{M \cap V_{x^0,u^0}} \frac{f(z,u^0) h^j (z,u^0) dx}{(x_1 - x_1^0)^n},$$

Thus, we analytically continue the function f onto a larger set M_1. Repeating the method again, we obtain the extension to M_2 with the property: if $(z^0,u^0) \in S$ and $V_{x^0,u^0} \cap M$ has positive measure m_k, then the points $(x+iy^0,u^0) \in M_2$, where $x \in \mathrm{R}^n$.

Now let (z^0,u^0) be an arbitrary point on S. Consider the second-class line $l_{b,p}$. If $F_1(b,b) \neq 0$, then we define:

$$p = \left[F_1(b,u^0) + i/2 \right] \left[F_1(b,b) \right]^{-1},$$

so that $z_1 = z_1^0 + t$ for $z \in l_{b,p}$.

Let $M_b = M_2 \cap l_{b,p}$, $M^{'} = \left\{ b:\ b \in C^m,\ m_1 M_b > 0 \right\}$. The latter set has positive $2m$-dimensional Lebesgue measure.

Further, we have:

$$dz_1 \wedge du \wedge d\bar{u} = |t|^{2m} dt \wedge db \wedge d\bar{b},$$

and:

$$\left[(z_1 - z_1^0) \left| z_1 - z_1^0 \right|^{2m} \right]^{-1} \partial z_1 \wedge \partial u \wedge \partial \bar{u} = t^{-1} dt \wedge db \wedge d\bar{b}.$$

Now we can recover the value of the function $f(at+z^0, bt+u^0)$ at $t=0$ using Theorem 4.2. Then, multiplying the result by $dz_1 \wedge d\bar{u}$ and integrating over M', we obtain the formula:

$$f(z^0,u^0) = \frac{1}{2\pi i \int\limits_{M^{'}} db \wedge d\bar{b}} \lim_{j\to\infty} \int\limits_{M_3} f(z,u) \frac{\varphi^j (z,u) dz_1 \wedge du \wedge d\bar{u}}{(z_1 - z_1^0) \left| z_1 - z_1^0 \right|^{2m}},$$

where M_3 is the projection of the set M_2 to the space of variables (z_1,u).

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