EduVision: Journal of Innovations in Pedagogy and Educational Advancements
Volume 2, Issue 6, June 2026
brightmindpublishing.com
ISSN (E): 3061-6972

THE TEYLOR SERIES AND THEIR PROPERTIES

Jumaniyazova Durdona Tuliboy kizi
Lecturer at the Department of Economics at
Tashkent University of Economics and Technology
G-mail: jdurdona.93@gmail.com

Abstract
This article examines Taylor series and their properties. Mainly, basic concepts related to convergence, such as the radius of convergence of power series, Abel's theorem, Cauchy-Adamard formula are analyzed. The uniqueness of the Taylor series, the methods of expanding functions into a Taylor series, and the properties of holomorphic functions are shown. The absolute and uniform convergence of series, the Cauchy inequality, and their significance in complex analysis are discussed. The article also examines the applications of Taylor series in mathematical analysis and complex analysis.

Keywords: Taylor series, convergent series, absolute convergence, Weierstrass criterion, Abel's theorem, Cauchy-Adamard formula, holomorphic function.

Introduction

In mathematical analysis and complex analysis, representing functions through series is of great importance. Taylor series are one of the most widely used methods in these fields, allowing functions to be expressed in simple form. This article provides an in-depth study of Taylor series and their properties. In particular, basic concepts related to convergence, such as the radius of convergence of power series, Abel's theorem, and the Cauchy-Adamard formula, are analyzed. The uniqueness of the Taylor series, the methods of expanding functions into a Taylor series, and the properties of holomorphic functions are shown. The absolute and uniform convergence of series, the Cauchy inequality, and their significance in complex analysis are discussed. The article also examines the applications of Taylor series in mathematical analysis and complex analysis. The research results offer new approaches that can be applied in the fields of mathematical modeling, physics, and engineering.

EduVision: Journal of Innovations in Pedagogy and Educational Advancements
Volume 2, Issue 6, June 2026
brightmindpublishing.com
ISSN (E): 3061-6972

MAIN PART

Power series

Definition 1.1.1. This

$$\sum_{n=0}^{\infty} c_n z^n \qquad (1.1.1)$$

or

$$\sum_{n=0}^{\infty} c_n (z-a)^n \qquad (1.1.2)$$

A series of the form is called a power series.
Complex numbers are called the coefficients of a power series.
If in (1.1.2) we say $z-a=\xi$, then the series (1.1.2) comes to the series of the form (1.1.1). Therefore, studying the series of the form (1.1.1) is sufficient.

1.1.1-theorem. If the power series (1.1.1) converges for some $(z \neq 0)$ values of z and diverges for some values, then there exists a unique number R (R>0) such that the series (1)

$$\{z \in \mathbb{C} : |z| < R\}$$

converging in the circle,

$$\{z \in \mathbb{C} : |z| > R\}$$

and in the field, it will be divergent.

Definition 1.1.2. The number R is called the radius of convergence of the power series (1.1.1), and the circle is called the circle of convergence of the power series, if the power series (1.1.1) converges and diverges at .

Note. (1.1.1) power series

$$\{z \in \mathbb{C} : |z| = R\}$$

points of the circle can either converge or diverge.

Properties:

1. If the radius of convergence of the power series (1.1.1) is R (R>0), then this series is .

$$\{z \in \mathbb{C} : |z| \le R_1, R_1 < R\}$$

is uniformly convergent in the circle.

EduVision: Journal of Innovations in Pedagogy and Educational Advancements
Volume 2, Issue 6, June 2026
brightmindpublishing.com
ISSN (E): 3061-6972

Proof. Since the radius of convergence of the given power series is R, the series

$\{z \in \mathbb{C} : |z| < R\}$

will converge in the circle.

$z_0 \in \{z \in \mathbb{C} : |z| \le R_1, R_1 < R\}$

let's take the point. Obviously, at this point, the power series converges absolutely, i.e.,

$$\sum_{n=0}^{\infty} |c_n z_0^n|$$

the series will converge.

$\forall z \in \{z \in \mathbb{C} : |z| \le |z_0|\}$

always

$$|c_n z^n| \le |c_n||z_0^n| = |c_n z_0^n|$$

according to the Weierstrass sign

$$\sum_{n=0}^{\infty} c_n z^n$$

is uniformly convergent in the series.

Result. Summary of series of degree (1.1.1)

$\{z \in \mathbb{C} : |z| \le R_1, R_1 < R\}$

has a continuous function.

2. If the degree (1.1.1) is the radius of convergence of the series R (R>0) , then this series can be derived term by term in .

Theorem 1.1.2 (Abel) . If

(1.1.3)

If the series of degree is convergent at the value of , then this series

$\{z \in \mathbb{C} : |z| < |z_0|\}$

will be absolutely convergent in the circle.

Proof. According to the condition

$$\sum_{n=0}^{\infty} c_n z_0{}^n \quad (1.1.4)$$

EduVision: Journal of Innovations in Pedagogy and Educational Advancements
Volume 2, Issue 6, June 2026
brightmindpublishing.com
ISSN (E): 3061-6972

the numerical series converges. By the necessary condition of series convergence

$$\lim_{n\to\infty} c_n z_0^n = 0$$

will be.

Since the sequence has a finite limit, then this sequence is bounded, i.e., there exists a constant number M>0 such that for

$$\left|c_n z_0^n\right| \le M$$

from this

$$\left|c_n z^n\right| = \left|c_n z_0^n\right| \cdot \left|\frac{z}{z_0}\right|^n \le M\left|\frac{z}{z_0}\right|^n \qquad (1.1.5)$$

now this

$$\sum_{n=0}^{\infty}\left|c_n z^n\right|$$

along with the following row

$$\sum_{n=0}^{\infty} M\left|\frac{z}{z_0}\right|^n$$

let's look at the row. Obviously

$$\sum_{n=o}^{\infty} M\left|\frac{z}{z_0}\right|^n$$

is convergent because $\left|\frac{z}{z_0}\right| = q < 1$, and according to (1.1.5), the series is convergent on the circle. Consequently, the given series converges absolutely within the range. The theorem was proven.

Result. If

$$\sum_{n=0}^{\infty} c_n z^n$$

is divergent at the point $z = z_1$, then the series is divergent in the domain.

Proof: Let the given power series be divergent at the point z = z. Then this series is convergent for the values of z that satisfy the inequality , since if the series is

EduVision: Journal of Innovations in Pedagogy and Educational Advancements
Volume 2, Issue 6, June 2026
brightmindpublishing.com
ISSN (E): 3061-6972

convergent for some value of z that satisfies the inequality , then according to Abel's theorem, this series becomes convergent at the point z = z, and at the point . This contradicts the statement that the series is divergent at the point z = z.

Consequently, the given series is divergent in . The result has been proven.

The Cauchy-Adamard theorem.

This

$$\sum_{n=0}^{\infty} c_n z^n$$

Let a power series be given.

From the coefficients of this series, we construct the following sequence of numbers: (1.1.6)

Let us denote the maximum limit of this sequence as L, i.e.

(1.1.7)

The radius of convergence of a series of degree is determined by the formula. This formula is called the Cauchy-Adamard formula.

Proof. a) Let $L = 0$, in which case it must be shown that the power series converges at any point. In this case, the sequence (1.1.6) approaches zero, so at the point and at this

$$\sqrt[n]{|c_n|} < \frac{\theta}{|z_0 - a|}$$

the inequality is valid starting from any large values of n. So, starting from some n,

$$|c_n(z_0 - a)^n| < \theta^n$$

can be. It follows from this that the power series converges absolutely at any point z.

b). In this case, it must be shown that the power series is divergent at any point other than the point z=a. Since the greatest limit of the sequence (1.1.6) is infinite, an infinite number of values of n are found for the point, and for these

$$\sqrt[n_k]{|c^{n_k}|} > \frac{1}{|z - a|}$$

the inequality is fulfilled. From this

$$|c_{n_k}(z - a)^{n_k}| > 1$$

EduVision: Journal of Innovations in Pedagogy and Educational Advancements
Volume 2, Issue 6, June 2026
brightmindpublishing.com
ISSN (E): 3061-6972

and this is a necessary condition for the power series to converge

$$\lim_{n\to\infty} c_n(z-a)^n = 0$$

will not be performed at any point.

v) L is a finite number, i.e. In this case, it must be shown that the power series converges at points satisfying the inequality and diverges at points satisfying the inequality.

Let it be . Since L is the largest limit, based on the property of this limit, starting from n

$$\sqrt[n]{|c_n|} < L+\varepsilon,\ \varepsilon>0.$$

Let's choose such that it satisfies the following inequality:

.

Then the previous inequality can be written as:

$$\sqrt[n]{|c_n|} < L+\frac{1}{2}\left(\frac{1}{|z-a|}-L\right)=\frac{1+L|z-a|}{2|z-a|}$$

or

$$\sqrt[n]{|c_n|}\,|z-a| < \frac{1+L|z-a|}{2}=\theta<1$$

because . From this

This means that the power series converges absolutely on the circle.

Now

$$|z-a|>\frac{1}{L}$$

Let it be . Any less than the definition of L and for infinitely many values of n

$$\sqrt[n]{|c_n|} > L-\varepsilon$$

it will be. Let's select it like this:

$$\varepsilon<(L-\frac{1}{|z-a|}).$$

In this case

$$\sqrt[n]{|c_n|} > \mathrm{L}-(L-\frac{1}{|z-a|})=\frac{1}{|z-a|}$$

EduVision: Journal of Innovations in Pedagogy and Educational Advancements
Volume 2, Issue 6, June 2026
brightmindpublishing.com
ISSN (E): 3061-6972

or

$$|c_n||z-a|^n > 1.$$

Since the last inequality holds for infinitely many values of n, the term cannot approach zero at , i.e., the series is divergent. Thus, the Cauchy-Adamard formula is proved for all cases.

Taylor's line. Let's say

$$\sum_{n=0}^{\infty} c_n (z-z_0)^n = c_0 + c_1(z-z_0) + c_2(z+z_0)^2 + \ldots + c_n(z-z_0)^n + \ldots$$

a power series is given, its radius of convergence). Obviously, this series

$$\{z \in \square : |z-z_0| < R\}$$

will converge in the circle. Assume the sum of a given power series:

(1.1.8)

Using the property of the power series mentioned above, we sequentially differentiate the series (1.1.8):

$$f'(z) = c_1 + c_2 2(z - z_0) + c_3 3(z - z_0)^2 + \mathrm{K} + c_n n(z - z_0)^{n-1} + \mathrm{K},$$

$$f''(z) = c_2 2 + c_3 3(z - z_0) + \mathrm{K} + c_n n(n - 1)(z - z_0)^{n-2} + \mathrm{K},$$

....

Let us assume that in these equalities.

$f(z_0) = c_0,$

$f'(z_0) = c_1,$

$f''(z_0) = 2!c_2,$

$f'''(z_0) = 3!c_3,$

....

$f^{(n)}(z_0) = n!c_n,$

....

so,

$$c_0 = f(z_0),\ c_1 = \frac{f'(z_0)}{1!},\ c_2 = \frac{f''(z_0)}{2!}, \mathrm{K}, c_n = \frac{f^{(n)}(z_0)}{n!}, \mathrm{K}$$

bo`ladi.

So,

EduVision: Journal of Innovations in Pedagogy and Educational Advancements
Volume 2, Issue 6, June 2026
brightmindpublishing.com
ISSN (E): 3061-6972

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n$$

The coefficients of a power series are expressed through the point values of the function and its derivatives. Substituting these coefficient values into (1.1.8).

(1.1.9)

can be. The power series (1.1.9) is commonly called the Taylor series.

Expand function to Taylor series. If , then the function at the point (around the point)

$$U_\rho(a)=\{z \in \mathbf{C}_z : |z-a| < \rho, \rho > 0\} \subset D$$

Taylor expands to:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n .$$

Proof. Let be the limit of . will be. First

$$\frac{1}{\xi - z} = \frac{1}{\xi - a - (z-a)} = \frac{1}{(\xi - z)\left(1 - \frac{z-a}{\xi - a}\right)}$$

then

$$\frac{1}{1 - \frac{z-a}{\xi - a}} = \sum_{n=0}^{\infty} \left(\frac{z-a}{\xi - a}\right)^n$$

let's find by taking into account:

$$\frac{1}{\xi - z} = \sum_{n=0}^{\infty} \left(\frac{z-a}{\xi - a}\right)^n \quad (1.1.10)$$

this is a geometric series, its denominator

$$\frac{z-a}{\xi - a}$$

ga teng.

Clearly, for $\xi \in \gamma$, the following inequality

$$\left|\frac{z-a}{\xi - a}\right| = \frac{|z-a|}{\rho} = q < 1$$

appropriate. (1.1.10) both sides of the equality

EduVision: Journal of Innovations in Pedagogy and Educational Advancements
Volume 2, Issue 6, June 2026
brightmindpublishing.com
ISSN (E): 3061-6972

$$\frac{1}{2\pi i} f(\xi)$$

and then integrating along the line γ, we obtain the following

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f(\xi)}{\xi - z}d\xi = \frac{1}{2\pi i}\int_{\gamma}\sum_{n=0}^{\infty}\frac{f(\xi)}{(\xi - a)^{n+1}}(z-a)^n d\xi$$

we come to equality. From the above relationship

(1.1.11)

it follows. Subintegral

$$\sum_{n=0}^{\infty}\frac{f(\xi)}{(\xi - a)^{n+1}}(z-a)^n$$

for the terms of the series

$$\left|\frac{f(\xi)}{(\xi - a)^{n+1}}(z-a)^n\right| < \frac{1}{\rho}Mq^n \quad (n = 1,2,\ldots)\left(M = \max_{\gamma}|f(\xi)|\right)$$

inequality holds. Obviously, the series converges.

In that case, according to the Weierstrass criterion

$$\sum_{n=0}^{\infty}\frac{f(\xi)}{(\xi - a)^{n+1}}(z-a)^n$$

the functional series γ is uniformly convergent. Consequently, this series can be integrated term by term. Then the equality (1.1.11) is

(1.1.12)

appears. According to the aforementioned theorem

(1.1.13)

let's find out. As a result, from equations (1.1.12) and (1.1.13),

$$f(\xi) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = \sum_{n=0}^{\infty}c_n(z-z_0)^n$$

. This means that the function $f(z)$ is expanded into a Taylor series.

Koshi inequality. If the function is holomorphic on a closed circle, and the boundary of this circle is on the circle , then the function for the coefficients of the Taylor series

(1.1.14)

inequality will be true.

EduVision: Journal of Innovations in Pedagogy and Educational Advancements
Volume 2, Issue 6, June 2026
brightmindpublishing.com
ISSN (E): 3061-6972

Hologomorphism of the series sum. Cauchy's integral formula for a derivative of arbitrary order. In this point, we will present several properties of a holomorphic function using the theorem on the holomorphism of the sum of a power series.

Theorem 1.1.3. The derivative of any function is also holomorphic in the domain.

Proof. First, let's construct a circle for the point. According to Taylor's theorem, this function is convergent in this domain

$$f(z) = \sum_{n=1}^{\infty} c_n (z-a)^n$$

expands to a power series. It is known from the previous point that this function has a derivative.

$$f' = \varphi = c_1 + 2c_2(z-a) + \ldots = \sum_{n=1}^{\infty} n c_n (z-a)^{n-1}$$

the function also represents a power series converging within the same range.

According to the theorem on the holomorphism of the sum of a power series, a function is holomorphic in . From this, i.e., from the arbitrariness of the point. The theorem has been proven.

Immediately from this theorem, for the antiderivative of a function of a complex variable, the following result follows.

Result. If a function that is continuous in a domain has an antiderivative in this domain, then it is.

Furthermore, if we repeatedly apply the above theorem to the derivative of a function, the following theorem will be valid in the general case.

Theorem 1.1.4. Any derivative of an arbitrary function is also a holomorphic function in a domain.

The following theorem states that the power series (Taylor series) of a function around a given point is unique.

Theorem 1.1.5. If the function is in the range

(1.1.15)

is expanded into a power series, then the coefficients of this series are as follows:

(1.1.16)

is uniquely determined using the formula.

Proof. It follows that if we say initially in the expansion (1.1.15), then . In the next step, we differentiate relation (1.1.15) by terms:

BRIGHT MIND PUBLISHING
EduVision: Journal of Innovations in Pedagogy and Educational Advancements
Volume 2, Issue 6, June 2026
brightmindpublishing.com
ISSN (E): 3061-6972

$$f'(z) = c_1 + 2c_2(z - z_0) + \ldots$$

and if we say here too, we find that. Continuing this process, i.e., if we differentiate relation (1.1.15) times, we obtain:

$$f^{(n)}(z) = n!c_n + c_1'(z - z_0) + \ldots$$

and if we say here too, we find that is. The theorem has been proven.

Note. This theorem implies that any convergent series is a Taylor series of its sum.

This is determined by formula (1.1.16), which is written in the previous paragraph

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(x)}{(x - a)^{n+1}} dx$$

by comparing the formulas, we obtain the following formula for the derivative of any order of a holomorphic function:

. (1.1.17)

Since this point is arbitrary, we obtain the following Cauchy integral formula for any derivative of a holomorphic function.

Theorem 1.1.6. If , then the function has a derivative of any order in and

(1.1.18)

the formula holds true.

References

1. Shabat B.V. Introduction to Comprehensive Analysis.ch.1,2. Moscow: "Nauka," 1985.
2. Sadullaev A. Multi-argument holomorphic functions. Urgench, 2004.
3. Zacharyuta V.P. Separate-analytic functions, generalizations of Hartog's theorem and holomorphic shells // Math. collection. 1976, vol. 101 (143), No1, pp. 57-76.
4. Садуллаев А. Плюрисубгармонические функции // Современные проблемы математики. Fundamental Directions. - M., VINITI, 1985, vol. 8, pp. 65-113. Results of Science and Technology of the VINITI of the USSR Academy of Sciences.
5. Садуллаев А., Чирка Е.М. О продолжении функций с полярными особенностями //Мат.сб. 1987, т. 132, No3, с. 383-390.

EduVision: Journal of Innovations in Pedagogy and Educational Advancements
Volume 2, Issue 6, June 2026
brightmindpublishing.com
ISSN (E): 3061-6972

6. Sadullaev A.S., Tuychiev T.T. On the extension of Hartog's series allowing holomorphic extension to parallel sections. // Uzb. Math. Journal. 2009, No. 1, p. 148-157.
7. Sadullaev A. Pluripotential Theory. Application. Palmarium academic publishing. Saarbrüchen Deutschland.