EduVision: Journal of Innovations in Pedagogy and Educational Advancements
Volume 01, Issue 04, April, 2025
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TRIGONOMETRY IN A GALILEAN PLANE

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Abstract
The parabolic trigonometry in the Galilean plane is introduced in analogy to the simple and hyperbolic trigonometries in the Euclidean and Minkowski planes. In this paper we will introduce parabolic functions and their properties in the Galilean plane.

Introduction

1.Preliminaries

Trigonometry (Greek: trigon - triangle and "metrezis" - measure) is a branch of mathematics that studies the properties of trigonometric functions, the relationship between the sides and angles of a triangle. Theorems of sines, cosines and tangents are the main results of trigonometry. Trigonometry has a wide range of practical applications. Trigonometry plays an important role in mathematics. Trigonometry was founded in the works of Muhammad ibn Musa al-Khorezmi, Al-Beruni, Al-Battani, Abu al-Wafa, Nasiriddin Tusi. And it was developed by Regiomontanus, the current designations and writing style were formed in the works of L. Euler.[6]
This paper is aimed at filling a gap in the theory of elementary functions, we will introduce indeed the trigonometric parabolic function and study the relevant properties. The hyperbolic functions can be viewed as parametric functions, through the parameter Φ.
The link between hyperbolic and trigonometric functions is usually done using the imaginary unit, but it will be more useful for our purposes, the introduction of the so called Gudermann function gd(Φ)[1], whose role can be understood as follows. According to Fig. 1 the relation between the parameter Φ and the corresponding angle φ is provided by

$$tg(\varphi) = tgh(\Phi) \qquad (1)$$

so that the Gudermann function writes

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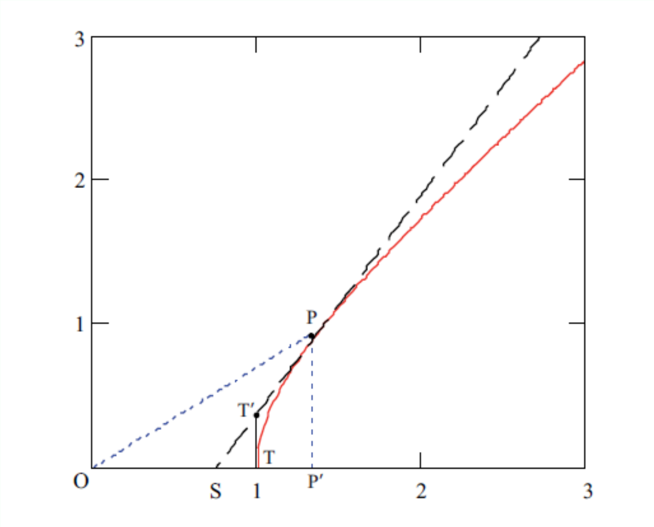

Fig. 1

$$\varphi = gd(\Phi) = tg^{-1}\left[tgh(\Phi)\right] \quad (2)$$

by taking the derivative with respect to Φ , we obtain the relation

$$\frac{d}{d\Phi} gd(\Phi) = \sec h(2\Phi) \quad (3)$$

which yields the alternative definition

$$gd(\Phi) = \int_0^{\Phi} \sec h(\sigma) d\sigma \quad (4)$$

We can take advantage from the above definition to get a deeper insight in the theory of generalized trigonometric functions. Ferrari proposed in [2] the following definition (n is any positive integer)

$$C(\Phi|n)^n + S(\Phi|n)^n = 1 \quad (5)$$

with the parameter Φ specified by

$$\frac{1}{2} C(\Phi|n) S(\Phi|n) + \int_{C(\Phi|n)}^{1} \left[1-\xi^n\right]^{\frac{1}{n}} d\xi = \frac{1}{2}\Phi \quad (6)$$

The equations (5,6) can be exploited to state the relevant properties under derivation

$$\frac{d}{d\Phi} C(\Phi|n) = -S(\Phi|n)^{n-1} \quad (7)$$

$$\frac{d}{d\Phi} S(\Phi|n) = C(\Phi|n)^{n-1}$$

Which, along with the conditions

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$$C(0|n)=1 \qquad S(0|n)=0 \tag{8}$$

can be exploited to derive the associated series expansions, as more carefully discussed below.

We will consider the following generalization of equations (5,6), with p,q two distinct integers

$$C(\Phi|p,q)^p + S(\Phi|p,q)^q = 1$$

$$\frac{1}{2}C(\Phi|p,q)S(\Phi|p,q) + \int_{C(\Phi|p,q)}^{1}\left[1-\xi^p\right]^{\frac{1}{q}} d\xi = \frac{1}{2}\Phi \tag{9}$$

which, once combined, yield the derivation rules

$$\frac{d}{d\Phi}C(\Phi|p,q) = -q\frac{S(\Phi|p,q)^{q-1}}{qS(\Phi|p,q)^q + pC(\Phi|p,q)^p}$$

$$\frac{d}{d\Phi}S(\Phi|p,q) = p\frac{C(\Phi|p,q)^{p-1}}{qS(\Phi|p,q)^q + pC(\Phi|p,q)^p}$$

(10)

It is also proved by direct check that

$$\frac{d}{d\Phi}T(\Phi|p,q) = \frac{1}{C(\Phi|p,q)^2} \tag{11}$$

$$T(\Phi|p,q) = \frac{S(\Phi|p,q)}{C(\Phi|p,q)}$$

In the following we will specialize the previous general results to particular values of the integers p,q.

2.The Parabolic Functions: Elementary properties in the Euclidian plane.

We will identify the parabolic trigonometric functions with

$$C(\Phi|2,1) = \cos p(\Phi) \qquad S(\Phi|2,1) = \sin p(\Phi) \tag{12}$$

$$\cos p(0) = 1 \quad \sin p(0) = 0$$

they satisfy the fundamental relation

$$cos^2 p(\varphi) + \sin p(\varphi) = 1 \tag{13}$$

Where,

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$$\cos p(\varphi) = \sqrt[3]{\frac{4-3\varphi}{2} + \sqrt{1+\frac{(4-3\varphi)^2}{4}}} - \sqrt[3]{\sqrt{1+\frac{(4-3\varphi)^2}{4}} - \frac{4-3\varphi}{2}}$$

$$\sin p(\varphi) = -\sqrt[3]{\left[\frac{4-3\varphi}{2} + \sqrt{1+\frac{(4-3\varphi)^2}{4}}\right]^2} - \sqrt[3]{\left[\sqrt{1+\frac{(4-3\varphi)^2}{4}} - \frac{4-3\varphi}{2}\right]^2} + 3$$

It is known that there are simple and hyperbolic trigonometries in the Euclidean and Minkowski planes. The main identity of this trigonometry is as follows:

$\sin^2\alpha + \cos^2\alpha = 1$ in the Euclidean plane,

$ch^2 x - sh^2 x = 1$ in the Minkowski plane.[4]

Parabolic trigonometry in the plane was introduced by G. Dattoli, M. Migliorati, M. Quattromini and P. E. Ricci, in which the arc length of the parabola is taken as the angle measurement [3].

3.Trigonometry in the Galilean plane.

Now we introduce trigonometric functions in the Galilean plane. Since the circle is a parabola in the Galilean plane, parabolic trigonometry is used. [5] In this plane, we introduce trigonometry the angle in the Galilean sense.

Definition 1: The $pinh$ of the angle h is the ordinate which is generated intersection of the circle $x^2 + y = 1$ and the straight line which connecting the origin of the coordinates to the point which rotating the point (1;0) to the angle h.

Definition 2: The $posh$ of the angle h is the abscissa which is generated intersection of the circle $x^2 + y = 1$ and the straight line which connecting the origin of the coordinates to the point which rotating the point (1;0) to the angle h.

$pos(0) = 1 \qquad pin(0) = 0$.

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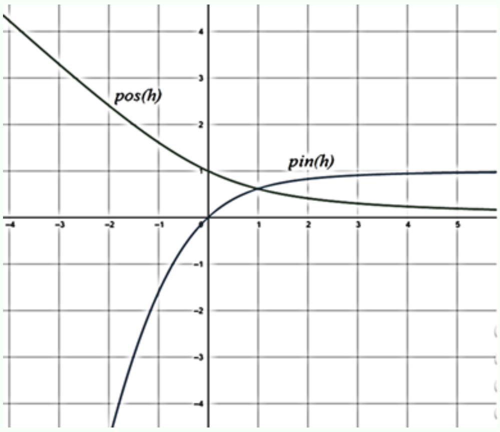

Fig. 2

In general trigonometry, it corresponds to the following situation, we look at this in the Galilean plane.

$$S(\varphi|2, 1) = pin(\varphi) \quad C(\varphi|2, 1) = pos(\varphi)$$

but since we now look at these equalities in the Galilean plane, the properties and differentials in general trigonometry will not be true in this trigonometry.

$$x^2 + y = 1$$

We get the point $M(x, y)$ from it. The projection of this point on the Ox axis is $pos(h)$, and its projection on the Oy axis is equal to $pin(h)$. $pos(h) = x \qquad pin(h) = y$.

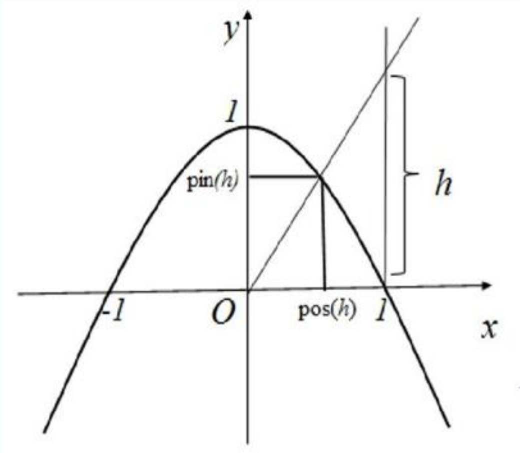

Fig. 3

$pos^2(h) + pin(h) = 1$ is valid.

$$pin(\infty) = 1 \ , \quad pos(\infty) = 0 .$$

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$$pos(h)=\frac{2}{\sqrt{h^2+4}+h} \qquad pin(h)=\frac{2h}{\sqrt{h^2+4}+h} \qquad (14)$$
$$ptg(h)=\frac{pin(h)}{pos(h)}=h \qquad ptg(h)\cdot cth(h)=1$$

In this trigonometry introduced in the Euclidean plane of trigonometry for $\forall m\in\square$, there is no revealing formula for determining $\sin(ma)$ and $\cos(ma)$ in terms of $\sin a$ and $\cos a$, but in the trigonometry of the Galilean plane, $pin(ah)$ and $pos(ah)$ can be determined, and they are as follows:

$$pos(\alpha\cdot h)=\frac{-\alpha\cdot pin(h)+\sqrt{\alpha^2\cdot pin^2(h)+4\cdot pos^2(h)}}{2} \qquad pin(\alpha\cdot h)=\alpha\cdot h\cdot pos(\alpha\cdot h)$$

We will give some properties of trigonometric functions in the Galilean plane:

- functions $pin(h)$ and $pos(h)$ are not periodic;
- functions $pin(h)$ and $pos(h)$ are neither odd nor even.

Theorem 1. For the trigonometric functions $pin(h)$ and $pos(h)$ in the Galilean plane, the following equations are valid:

$$\begin{cases} pin(h)+pin(-h)=-h^2 \\ pos(h)-pos(-h)=-h \\ pos(h)\cdot pos(-h)=1 \\ pin(h)\cdot pin(-h)=-h^2 \end{cases} . \qquad (15)$$

Proof: Let us assume equations (15) is valid for trigonometric functions. We know that if an equality is an identity, then by calculating the left side of the equality we can write the right side, and vice versa. Therefore, if we put the expressions in (14) on the left side of the equation (15), the left side of the equation will be equal to the right side of (15). And the reverse is also true. Now we proof these identities:

1) $pin(h)+pin(-h)=-h^2$

$$pin(h)=\frac{2h}{\sqrt{h^2+4}+h} \qquad pin(-h)=\frac{-2h}{\sqrt{h^2+4}-h}$$

$$pin(h)+pin(-h)=\frac{2h}{\sqrt{h^2+4}+h}+\frac{-2h}{\sqrt{h^2+4}-h}=\frac{2h\cdot(\sqrt{h^2+4}-h)-2h\cdot(\sqrt{h^2+4}+h)}{h^2+4-h^2}=\frac{-4h^2}{4}=h^2$$

2) $pos(h)-pos(-h)=-h$

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$$pos(h)=\frac{2}{\sqrt{h^2+4}+h} \qquad pos(-h)=\frac{2}{\sqrt{h^2+4}-h}$$

$$pos(h)-pos(-h)=\frac{2}{\sqrt{h^2+4}+h}-\frac{2}{\sqrt{h^2+4}-h}=\frac{2\cdot(\sqrt{h^2+4}-h)-2\cdot(\sqrt{h^2+4}+h)}{h^2+4-h^2}=\frac{-4h}{4}=-h$$

3) $pos(h)\cdot pos(-h)=1$

$$pos(h)=\frac{2}{\sqrt{h^2+4}+h} \qquad pos(-h)=\frac{2}{\sqrt{h^2+4}-h}$$

$$pos(h)\cdot pos(-h)=\frac{2}{\sqrt{h^2+4}+h}*\frac{2}{\sqrt{h^2+4}-h}=\frac{2\cdot 2}{h^2+4-h^2}=\frac{4}{4}=1$$

4) $pin(h)\cdot pin(-h)=-h^2$

$$pin(h)=\frac{2h}{\sqrt{h^2+4}+h} \qquad pin(-h)=\frac{-2h}{\sqrt{h^2+4}-h}$$

$$pin(h)\cdot pin(-h)=\frac{2h}{\sqrt{h^2+4}+h}\cdot\frac{-2h}{\sqrt{h^2+4}-h}=\frac{2h\cdot(-2h)}{h^2+4-h^2}=\frac{-4h^2}{4}=-h^2$$

Theorem is proved.

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